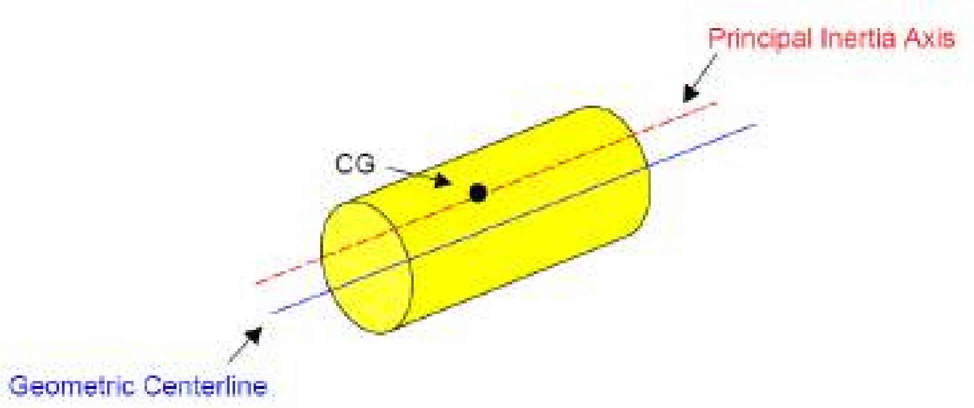
BALANCING OF ROTATING MASSES



Rotating centerline:
The rotating centerline being defined as the axis about which the rotor would rotate if not constrained by its bearings. (Also called the Principle Inertia Axis or PIA).

Geometric centerline:
The geometric centerline being the physical centerline of the rotor.

When the two centerlines are coincident, then the rotor will be in a state of balance. When they are apart, the rotor will be unbalanced.

Different types of unbalance can be defined by the relationship between the two centerlines. These include:
Static Unbalance – where the PIA is displaced parallel to the geometric centerline. (Shown above)
Couple Unbalance – where the PIA intersects the geometric centerline at the center of gravity. (CG)
Dynamic Unbalance – where the PIA and the geometric centerline do not coincide or touch.
The most common of these is dynamic unbalance.

Causes of Unbalance:
In the design of rotating parts of a machine every care is taken to eliminate any out of balance or couple, but there will be always some residual unbalance left in the finished part because of

1. slight variation in the density of the material or
2. inaccuracies in the casting or
3. inaccuracies in machining of the parts.

Why balancing is so important?

1. A level of unbalance that is acceptable at a low speed is completely unacceptable at a higher speed.
2. As machines get bigger and go faster, the effect of the unbalance is much more severe.
3. The force caused by unbalance increases by the square of the speed.
4. If the speed is doubled, the force quadruples; if the speed is tripled the force increases

by a factor of nine!

Identifying and correcting the mass distribution and thus minimizing the force and resultant vibration is very very important

BALANCING:

Balancing is the technique of correcting or eliminating unwanted inertia forces or moments in rotating or reciprocating masses and is achieved by changing the location of the mass centers.
The objectives of balancing an engine are to ensure:

1. That the centre of gravity of the system remains stationery during a complete revolution of the crank shaft and
2. That the couples involved in acceleration of the different moving parts balance each other.

Types of balancing:

a) **Static Balancing:**
i) Static balancing is a balance of forces due to action of gravity.
ii) A body is said to be in static balance when its centre of gravity is in the axis of rotation.

b) **Dynamic balancing:**
i) Dynamic balance is a balance due to the action of inertia forces.
ii) A body is said to be in dynamic balance when the resultant moments or couples, which involved in the acceleration of different moving parts is equal to zero.
iii) The conditions of dynamic balance are met, the conditions of static balance are also met.

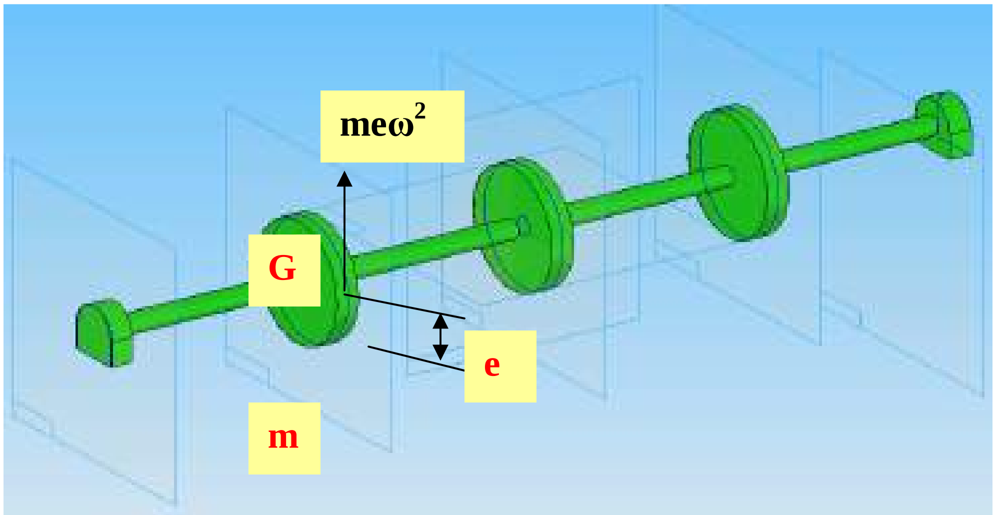
In rotor or reciprocating machines many a times unbalance of forces is produced due to inertia forces associated with the moving masses. If these parts are not properly balanced, the dynamic forces are set up and forces not only increase loads on bearings and stresses in the various components, but also unpleasant and dangerous vibrations.

Balancing is a process of designing or modifying machinery so that the unbalance is reduced to an acceptable level and if possible eliminated entirely.

BALANCING OF ROTATING MASSES

When a mass moves along a circular path, it experiences a centripetal acceleration and a force is required to produce it. An equal and opposite force called centrifugal force acts radially outwards and is a disturbing force on the axis of rotation. The magnitude of this remains constant but the direction changes with the rotation of the mass.

In a revolving rotor, the centrifugal force remains balanced as long as the centre of the mass of rotor lies on the axis of rotation of the shaft. When this does not happen, there is an eccentricity and an unbalance force is produced. This type of unbalance is common in steam turbine rotors, engine crankshafts, rotors of compressors, centrifugal pumps etc.



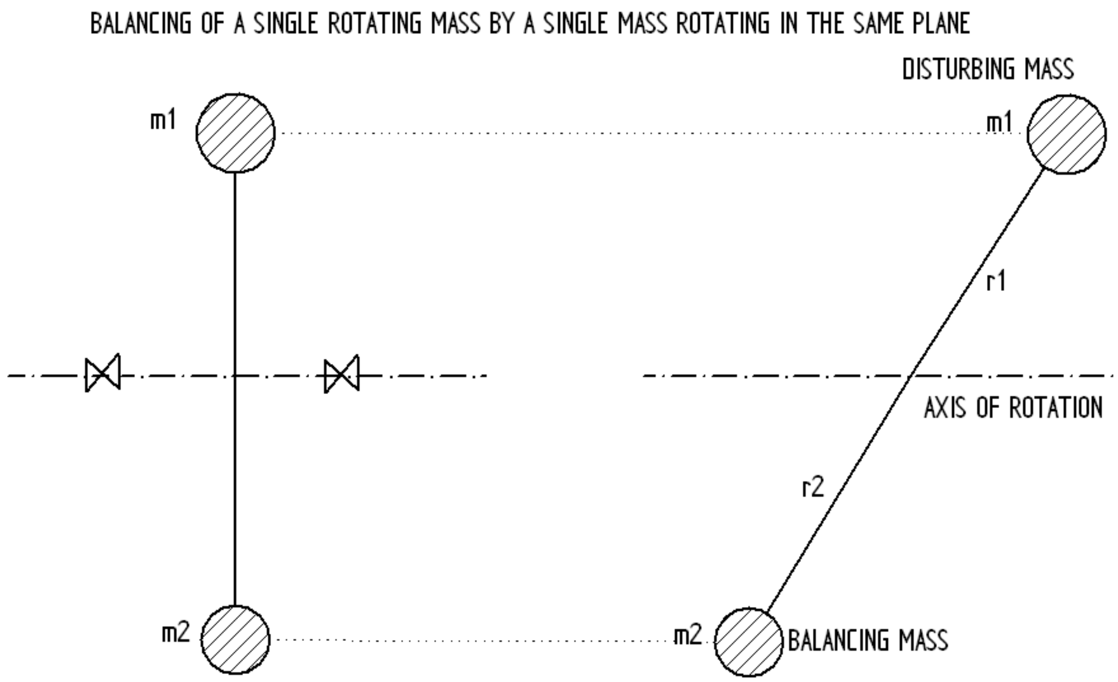
The unbalance forces exerted on machine members are time varying, impart vibratory motion and noise, there are human discomfort, performance of the machine deteriorate and detrimental effect on the structural integrity of the machine foundation.

Balancing involves redistributing the mass which may be carried out by addition or removal of mass from various machine members
Balancing of rotating masses can be of
1. Balancing of a single rotating mass by a single mass rotating in the same plane.
2. Balancing of a single rotating mass by two masses rotating in different planes.
3. Balancing of several masses rotating in the same plane
4. Balancing of several masses rotating in different planes

STATIC BALANCING:
A system of rotating masses is said to be in static balance if the combined mass centre of the system lies on the axis of rotation
DYNAMIC BALANCING;
When several masses rotate in different planes, the centrifugal forces, in addition to being out of balance, also form couples. A system of rotating masses is in dynamic balance when there does not exist any resultant centrifugal force as well as resultant couple.

CASE 1.

BALANCING OF A SINGLE ROTATING MASS BY A SINGLE MASS ROTATING IN THE SAME PLANE



Consider a disturbing mass m_1 which is attached to a shaft rotating at ω rad/s.
Let

$$\mathbf{r_1 = radius\ of\ rotation\ of\ the\ mass\ m_1}$$
$$\mathbf{= distance\ between\ the\ axis\ of\ rotation\ of\ the\ shaft\ and\ the\ centre\ of\ gravity\ of\ the\ mass\ m_1}$$

The centrifugal force exerted by mass m_1 on the shaft is given by,

$$\mathbf{F_{c1} = m_1\,\omega^2\,r_1} \text{ ------------------- } (1)$$

This force acts radially outwards and produces bending moment on the shaft. In order to counteract the effect of this force F_{c1} , a balancing mass m_2 may be attached in the same plane of rotation of the disturbing mass m_1 such that the centrifugal forces due to the two masses are equal and opposite.

Let,

$$\mathbf{r_2 = radius\ of\ rotation\ of\ the\ mass\ m_2}$$
$$\mathbf{= distance\ between\ the\ axis\ of\ rotation\ of\ the\ shaft\ and}$$
$$\mathbf{the\ centre\ of\ gravity\ of\ the\ mass\ m_2}$$

Therefore the centrifugal force due to mass m_2 will be,

$$\mathbf{F_{c2} = m_2\, \omega^2\, r_2 ------------------(2)}$$

Equating equations (1) and (2), we get

$$F_{c1} = F_{c2}$$
$$m_1\, \omega^2\, r_1 = m_2\, \omega^2\, r_2 \quad \text{or} \quad m_1\, r_1 = m_2\, r_2 -----------------(3)$$

The product $\mathbf{m_2\ r_2}$ can be split up in any convenient way. As for as possible the radius of rotation of mass m_2 that is r_2 is generally made large in order to reduce the balancing mass m_2.

CASE 2:
BALANCING OF A SINGLE ROTATING MASS BY TWO MASSES ROTATING IN DIFFERENT PLANES.

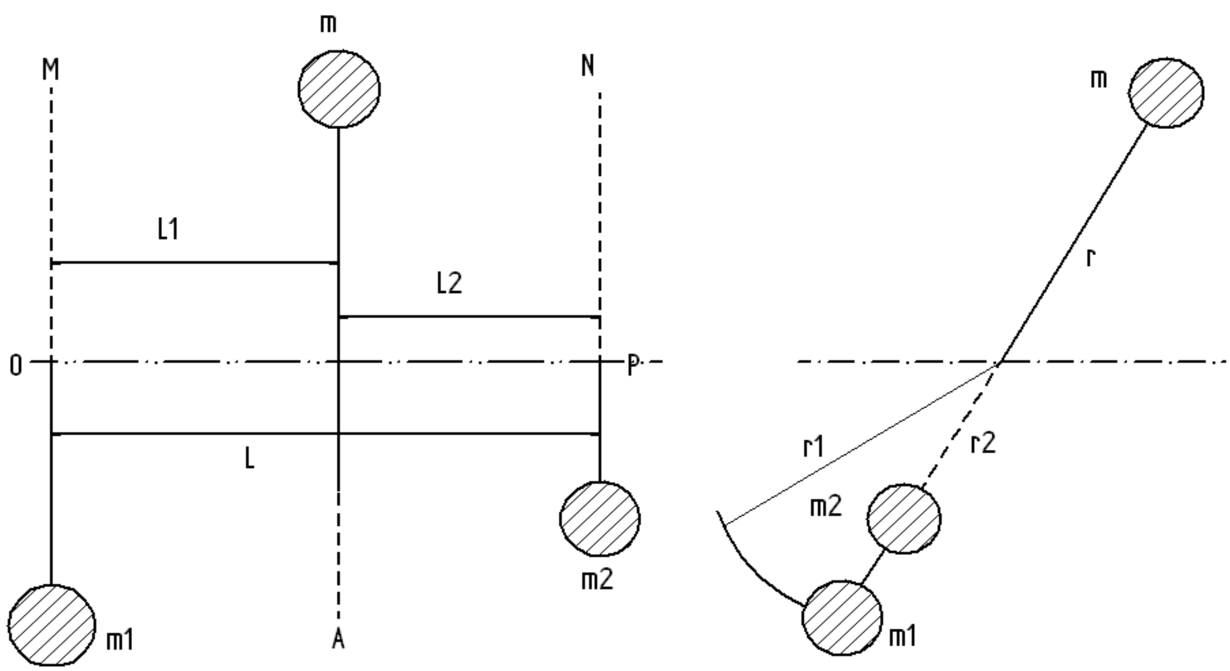
There are two possibilities while attaching two balancing masses:
1. The plane of the disturbing mass may be in between the planes of the two balancing masses.
2. The plane of the disturbing mass may be on the left or right side of two planes containing the balancing masses.

In order to balance a single rotating mass by two masses rotating in different planes which are parallel to the plane of rotation of the disturbing mass i) the net dynamic force acting on the shaft must be equal to zero, i.e. the centre of the masses of the system must lie on the axis of rotation and this is the condition for static balancing ii) the net couple due to the dynamic forces acting on the shaft must be equal to zero, i.e. the algebraic sum of the moments about any point in the plane must be zero. The conditions i) and ii) together give dynamic balancing.

CASE 2(I):

THE PLANE OF THE DISTURBING MASS LIES IN BETWEEN THE PLANES OF THE TWO BALANCING MASSES.

The plane of the disturbing mass lies inbetween the planes of the two balancing masses



Consider the disturbing mass m lying in a plane A which is to be balanced by two rotating masses m_1 and m_2 lying in two different planes M and N which are parallel to the plane A as shown.

Let r, r_1 and r_2 be the radii of rotation of the masses in planes A, M and N respectively.
Let L_1, L_2 and L be the distance between A and M, A and N, and M and N respectively.
Now,
The centrifugal force exerted by the mass m in plane A will be,

$$F_c = m\ \omega^2 r \text{ ------------------------(1)}$$

Similarly,
The centrifugal force exerted by the mass m_1 in plane M will be,

$$F_{c1} = m_1 \omega^2 r_1 \text{ ------------------------(2)}$$

And the centrifugal force exerted by the mass m_2 in plane N will be,

$$F_{c2} = m_2 \omega^2 r_2 ------------------(3)$$

For the condition of static balancing,

$$F_c = F_{c1} + F_{c2}$$
$$\text{or} \quad m\omega^2 r = m_1 \omega^2 r_1 + m_2 \omega^2 r_2$$
$$\text{i.e.} \quad m r = m_1 r_1 + m_2 r_2 ----------------(4)$$

Now, to determine the magnitude of balancing force in the plane 'M' or the dynamic force at the bearing 'O' of a shaft, take moments about ' P ' which is the point of intersection of the plane N and the axis of rotation.

Therefore,

$$F_{c1} x L = F_c x L_2$$
$$\text{or} \quad m_1 \omega^2 r_1 \; x \, L = m\omega^2 r \, x L_2$$
Therefore,
$$m_1 r_1 L = mrL_2 \quad \text{or} \quad m_1 r_1 = mr\frac{L_2}{L} --------(5)$$

Similarly, in order to find the balancing force in plane 'N' or the dynamic force at the bearing 'P' of a shaft, take moments about ' O ' which is the point of intersection of the plane M and the axis of rotation.

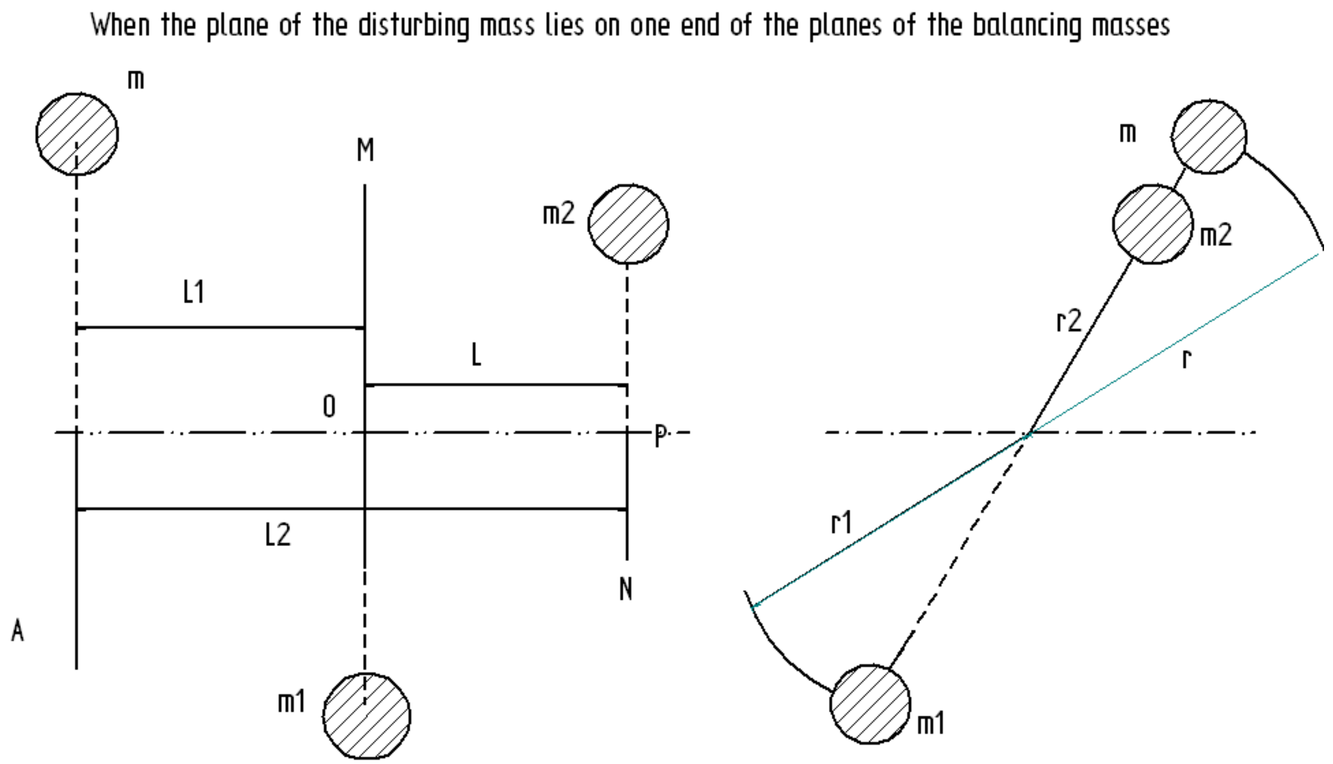
Therefore,

$$F_{c2} x L = F_c x L_1$$
$$\text{or} \quad m_2 \omega^2 r_2 \; x \, L = m\omega^2 r \, x L_1$$
Therefore,
$$m_2 r_2 L = mrL_1 \quad \text{or} \quad m_2 r_2 = mr\frac{L_1}{L} --------(6)$$

For dynamic balancing equations (5) or (6) must be satisfied along with equation (4).

CASE 2(II):

WHEN THE PLANE OF THE DISTURBING MASS LIES ON ONE END OF THE TWO PLANES CONTAINING THE BALANCING MASSES.



For static balancing,

$$F_{c1} = F_c + F_{c2}$$
$$\text{or} \quad m_1 \omega^2 r_1 = m\omega^2 r + m_2 \omega^2 r_2$$
$$\text{i.e.} \quad m_1 r_1 = m r + m_2 r_2 ----------------(1)$$

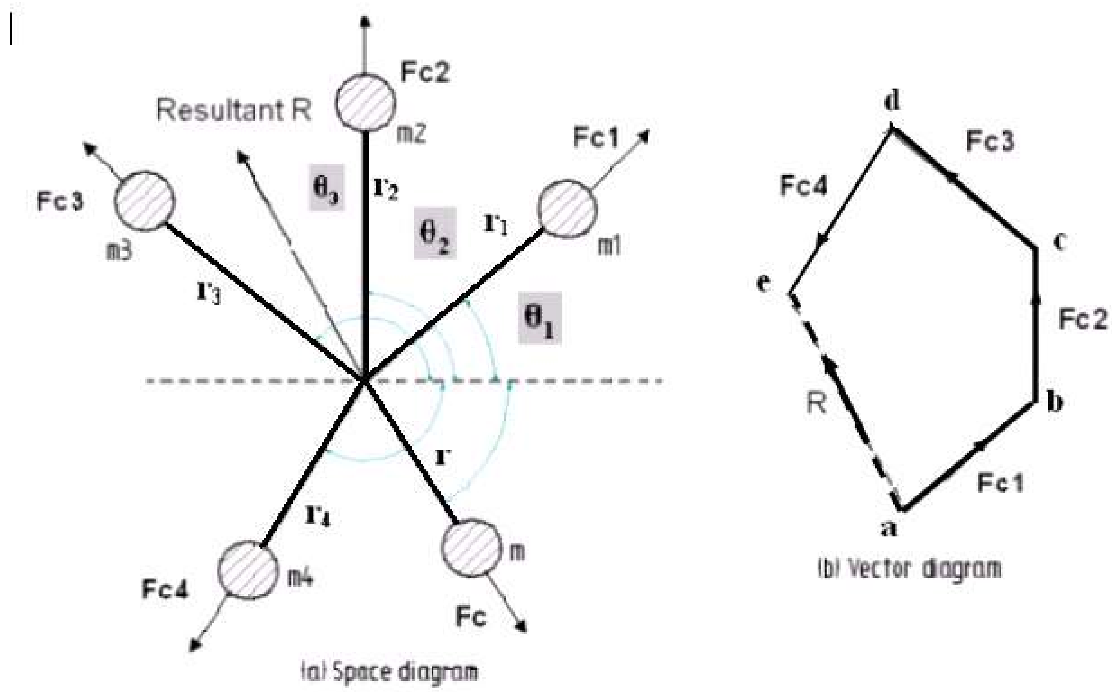
For dynamic balance the net dynamic force acting on the shaft and the net couple due to dynamic forces acting on the shaft is equal to zero.
To find the balancing force in the plane ‘M’ or the dynamic force at the bearing ‘O’ of a shaft, take moments about ‘P’. i.e.

$$F_{c1} \times L = F_c \times L_2$$
$$\text{or} \quad m_1 \omega^2 r_1 \times L = m\omega^2 r \times L_2$$
Therefore,
$$m_1 r_1 L = mrL_2 \quad \text{or} \quad m_1 r_1 = mr\frac{L_2}{L} ---------(2)$$

Similarly, to find the balancing force in the plane 'N' , take moments about 'O', i.e.,

$$F_{c2} \times L = F_c \times L_1$$
$$\text{or} \quad m_2 \omega^2 r_2 \times L = m\omega^2 r \times L_1$$
Therefore,
$$m_2 r_2 L = mrL_1 \quad \text{or} \quad m_2 r_2 = mr\frac{L_1}{L} ---------(3)$$

CASE 3:
BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE



BALANCING OF SEVERAL MASSES ROTATING IN THE SAME PLANE

Consider a rigid rotor revolving with a constant angular velocity ω rad/s. A number of masses say, four are depicted by point masses at different radii in the same transverse plane.

If m_1, m_2, m_3 and m_4 are the masses revolving at radii r_1, r_2, r_3 and r_4 respectively in the same plane.
The centrifugal forces exerted by each of the masses are F_{c1}, F_{c2}, F_{c3} and F_{c4} respectively.
Let F be the vector sum of these forces. i.e.

$$F = F_{c1} + F_{c2} + F_{c3} + F_{c4}$$
$$= m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 ----------(1)$$

The rotor is said to be statically balanced if the vector sum F is zero. If the vector sum F is not zero, i.e. the rotor is unbalanced, then introduce a counterweight (balance weight) of mass ‘m’ at radius ‘r’ to balance the rotor so that,

$$m_1 \omega^2 r_1 + m_2 \omega^2 r_2 + m_3 \omega^2 r_3 + m_4 \omega^2 r_4 + m \ \omega^2 r = 0 ----------(2)$$

or

$$m_1 r_1 + m_2 r_2 + m_3 r_3 + m_4 r_4 + m \ r = 0 -----------------(3)$$

The magnitude of either ‘m’ or ‘r’ may be selected and the other can be calculated.
In general, if $\sum \mathbf{m_i r_i}$ is the vector sum of $\mathbf{m_1 r_1}$, $\mathbf{m_2 r_2}$, $\mathbf{m_3 r_3}$, $\mathbf{m_4 r_4}$ etc, then,

$$\sum m_i r_i + mr = 0 ---------(4)$$

The above equation can be solved either analytically or graphically.

1. Analytical Method:

Procedure:
Step 1: Find out the centrifugal force or the product of mass and its radius of rotation exerted by each of masses on the rotating shaft, since ω^2 is same for each mass, therefore the magnitude of the centrifugal force for each mass is proportional to the product of the respective mass and its radius of rotation.
Step 2: Resolve these forces into their horizontal and vertical components and find their sums. i.e.,

$$\textbf{Sum of the horizontal components}$$
$$= \sum_{i=1}^{n} \mathbf{m_i r_i \cos\theta_i = m_1 r_1 \cos\theta_1 + m_2 r_2 \cos\theta_2 + m_3 r_3 \cos\theta_3} + ---------$$

$$\textbf{Sumof the vertical components}$$
$$= \sum_{i=1}^{n} \mathbf{m_i r_i \sin\theta_i = m_1 r_1 \sin\theta_1 + m_2 r_2 \sin\theta_2 + m_3 r_3 \sin\theta_3} + ---------$$

Step 3: Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

Step 4: If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i}$$

Step 5: The balancing force is then equal to the resultant force, but in opposite direction.
Step 6: Now find out the magnitude of the balancing mass, such that

$$R = mr$$

Where, m = balancing mass and r = its radius of rotation

2. Graphical Method:

Step 1:
Draw the space diagram with the positions of the several masses, as shown.

Step 2:
Find out the centrifugal forces or product of the mass and radius of rotation exerted by each mass.

Step 3:
Now draw the vector diagram with the obtained centrifugal forces or product of the masses and radii of rotation. To draw vector diagram take a suitable scale.
Let ab, bc, cd, de represents the forces F_{c1}, F_{c2}, F_{c3} and F_{c4} on the vector diagram.
Draw 'ab' parallel to force F_{c1} of the space diagram, at 'b' draw a line parallel to force F_{c2}. Similarly draw lines cd, de parallel to F_{c3} and F_{c4} respectively.

Step 4:
As per polygon law of forces, the closing side 'ae' represents the resultant force in magnitude and direction as shown in vector diagram.

Step 5:
The balancing force is then , equal and opposite to the resultant force.

Step 6:

Determine the magnitude of the balancing mass (m) at a given radius of rotation (r), such that,

$$F_c = m\omega^2 r$$

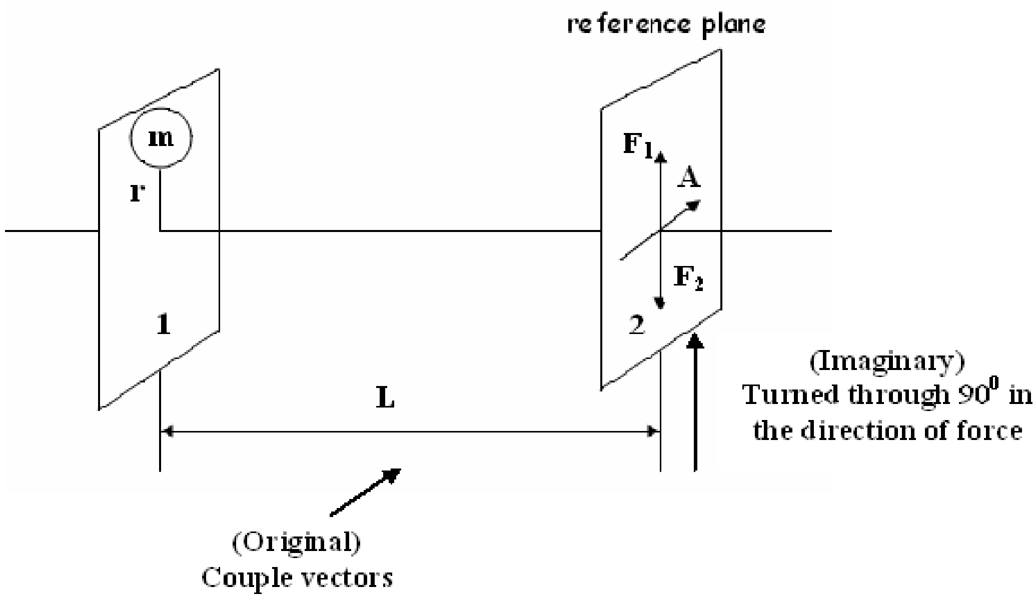
or

$$mr = \text{resultant of } m_1 r_1, m_2 r_2, m_3 r_3 \text{ and } m_4 r_4$$

CASE 4:

BALANCING OF SEVERAL MASSES ROTATING IN DIFFERENT PLANES

When several masses revolve in different planes, they may be transferred to a reference plane and this reference plane is a plane passing through a point on the axis of rotation and perpendicular to it.



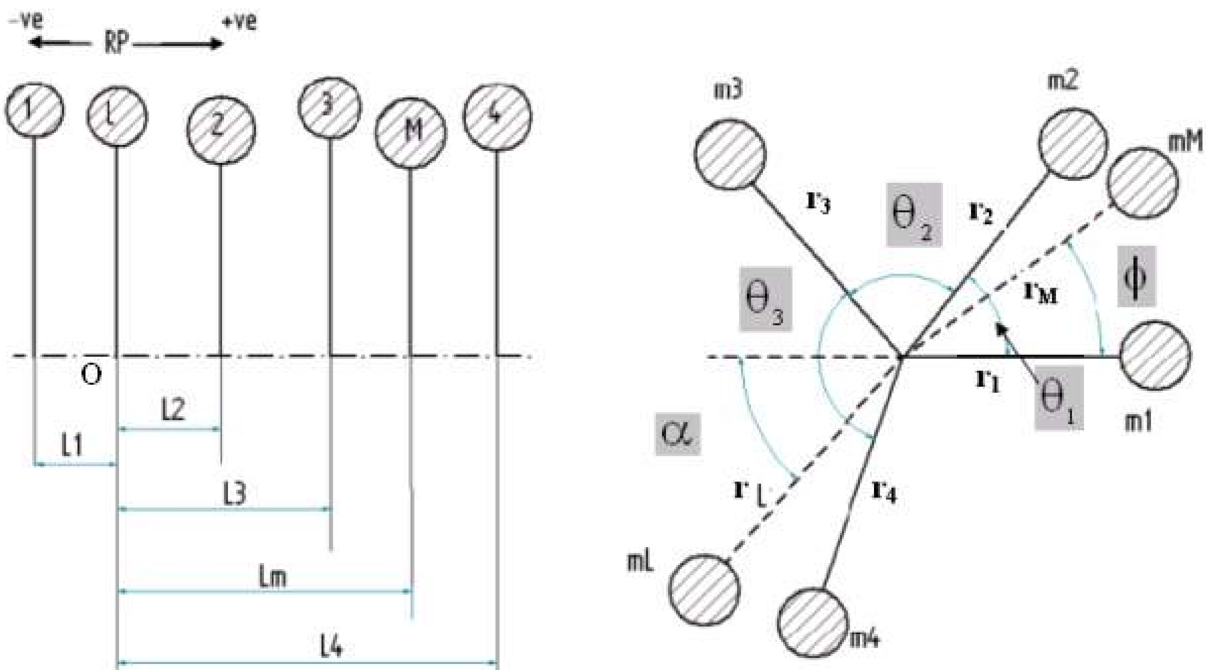
When a revolving mass in one plane is transferred to a reference plane, its effect is to cause a force of same magnitude to the centrifugal force of the revolving mass to act in the reference plane along with a couple of magnitude equal to the product of the force and the distance between the two planes.

In order to have a complete balance of the several revolving masses in different planes,

1. the forces in the reference plane must balance, i.e., the resultant force must be zero and
2. the couples about the reference plane must balance i.e., the resultant couple must be zero.

A mass placed in the reference plane may satisfy the first condition but the couple balance is satisfied only by two forces of equal magnitude in different planes. Thus, in general, two planes are needed to balance a system of rotating masses.

Example:
Consider four masses m_1, m_2, m_3 and m_4 attached to the rotor at radii r_1, r_2, r_3 and r_4 respectively. The masses m_1, m_2, m_3 and m_4 rotate in planes 1, 2, 3 and 4 respectively.



(a) position of planes of masses

(b) Angular position of masses

a) Position of planes of masses

Choose a reference plane at ‘O’ so that the distance of the planes 1, 2, 3 and 4 from ‘O’ are L_1, L_2 , L_3 and L_4 respectively. The reference plane chosen is plane ‘L’. Choose another plane ‘M’ between plane 3 and 4 as shown.

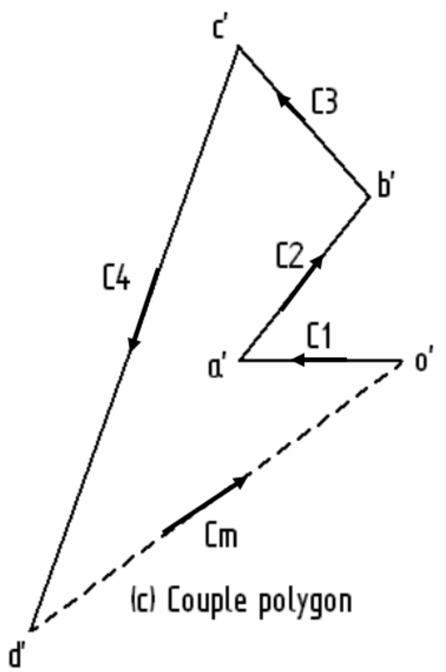
Plane ‘M’ is at a distance of L_m from the reference plane ‘L’. The distances of all the other planes to the left of ‘L’ may be taken as negative(-ve) and to the right may be taken as positive (+ve).

The magnitude of the balancing masses m_L and m_M in planes L and M may be obtained by following the steps given below.

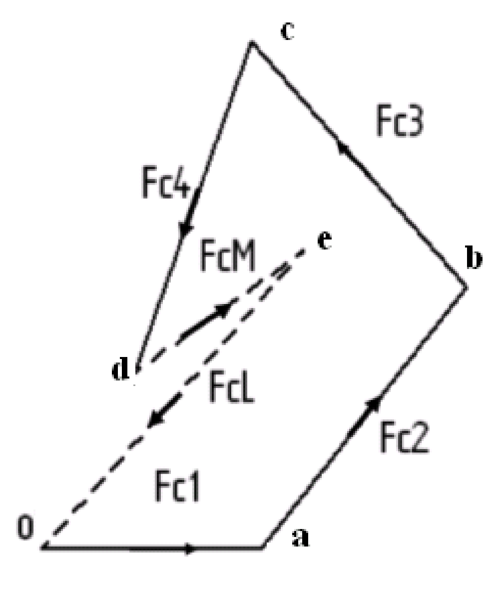
Step 1:
Tabulate the given data as shown after drawing the sketches of position of planes of masses and angular position of masses. The planes are tabulated in the same order in which they occur from left to right.

Plane 1	Mass (m) 2	Radius (r) 3	Centrifugal force/ω^2 (m r) 4	Distance from Ref. plane 'L' (L) 5	Couple/ ω^2 (m r L) 6
1	m_1	r_1	$m_1\ r_1$	$- L_1$	$- m_1\ r_1\ L_1$
L	m_L	r_L	$m_L\ r_L$	0	0
2	m_2	r_2	$m_2\ r_2$	L_2	$m_2\ r_2\ L_2$
3	m_3	r_3	$m_3\ r_3$	L_3	$m_3\ r_3\ L_3$
M	m_M	r_M	$m_M\ r_M$	L_M	$m_M\ r_M\ L_M$
4	m_4	r_4	$m_4\ r_4$	L_4	$m_4\ r_4\ L_4$

Step 2:
Construct the couple polygon first. (The couple polygon can be drawn by taking a convenient scale)
Add the known vectors and considering each vector parallel to the radial line of the mass draw the couple diagram. Then the closing vector will be '$m_M\ r_M\ L_M$'.



(c) Couple polygon



(d) Force polygon

The vector d 'o' on the couple polygon represents the balanced couple. Since the balanced couple C_M is proportional to $m_M\ r_M\ L_M$, therefore,

$$C_M = m_M\, r_M\, L_M = \text{ vector } d'o'$$

$$\text{or} \quad m_M = \frac{\text{vector } d'o'}{r_M\, L_M}$$

From this the value of m_M in the plane M can be determined and the angle of inclination ϕ of this mass may be measured from figure (b).

Step 3:
Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors along with '$m_M\, r_M$'. The closing vector will be '$m_L\, r_L$'. This represents the balanced force. Since the balanced force is proportional to '$m_L\, r_L$' ,

$$m_L\, r_L = \text{ vector } eo$$

$$\text{or} \quad m_L = \frac{\text{vector } eo}{r_L}$$

From this the balancing mass m_L can be obtained in plane 'L' and the angle of inclination of this mass with the horizontal may be measured from figure (b).

Problems and solutions

Problem 1.
Four masses A, B, C and D are attached to a shaft and revolve in the same plane. The masses are 12 kg, 10 kg, 18 kg and 15 kg respectively and their radii of rotations are 40 mm, 50 mm, 60 mm and 30 mm. The angular position of the masses B, C and D are 60^0 , 135^0 and 270^0 from mass A. Find the magnitude and position of the balancing mass at a radius of 100 mm.

Solution:
Given:

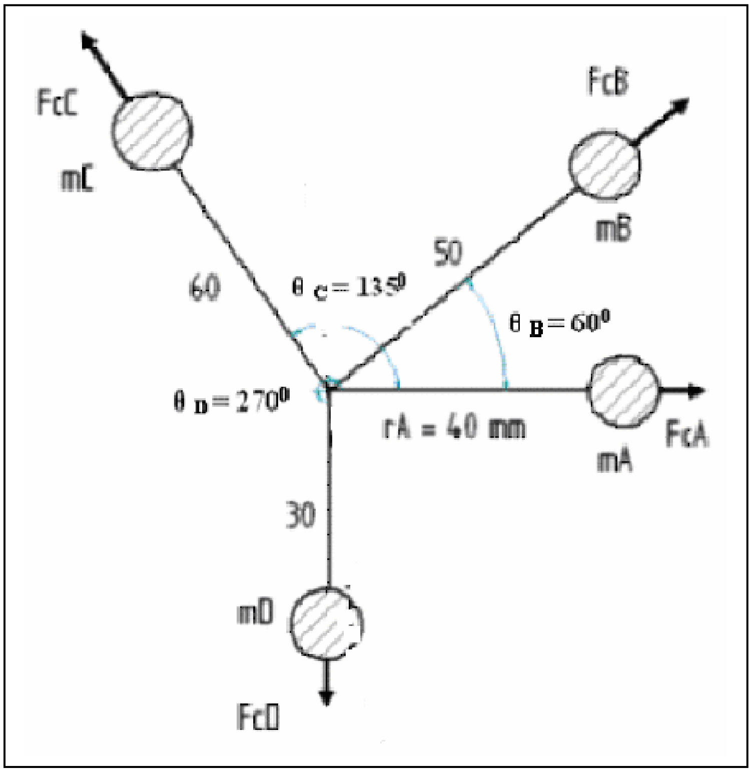
Mass(m) kg	Radius(r) m	Centrifugal force/ω^2 (m r) kg-m	Angle(θ)
m_A = 12 kg (reference mass)	r_A = 0.04 m	$m_A r_A$ = 0.48 kg-m	$\theta_A = 0^0$
m_B = 10 kg	r_B = 0.05 m	$m_B r_B$ = 0.50 kg-m	$\theta_B = 60^0$
m_C = 18 kg	r_C = 0.06 m	$m_C r_C$ = 1.08 kg-m	$\theta_C = 135^0$
m_D = 15 kg	r_D = 0.03 m	$m_D r_D$ = 0.45 kg-m	$\theta_D = 270^0$

To determine the balancing mass 'm' at a radius of r = 0.1 m.

The problem can be solved by either analytical or graphical method.

Analytical Method:

Step 1:
Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$.



Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

Step 2:
Resolve the centrifugal forces horizontally and vertically and find their sum.
Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$

$$= 0.48 \text{ x } \cos 0^0 + 0.50 \text{ x } \cos 60^0 + 1.08 \text{ x } \cos 135^0 + 0.45 \text{ x } \cos 270^0$$

$$= 0.48 + 0.25 + (-0.764) + 0 = -0.034 \text{ kg} - \text{m} \quad ---------(1)$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$

$$= 0.48 \text{ x } \sin 0^0 + 0.50 \text{ x } \sin 60^0 + 1.08 \text{ x } \sin 135^0 + 0.45 \text{ x } \sin 270^0$$

$$= 0 + 0.433 + 0.764 + (-0.45) = 0.747 \text{ kg} - \text{m} \quad ----------(2)$$

Step 3:
Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$

$$= \sqrt{(-0.034)^2 + (0.747)^2} = 0.748 \text{kg} - \text{m}$$

Step 4:
The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = mr = 0.748 \text{kg} - \text{m}$$

$$\text{Therefore,} \quad m = \frac{R}{r} = \frac{0.748}{0.1} = 7.48 \text{ kg Ans}$$

Where, m = balancing mass and r = its radius of rotation

Step 5:
Determine the position of the balancing mass 'm'.
If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{0.747}{-0.034} = -21.97$$

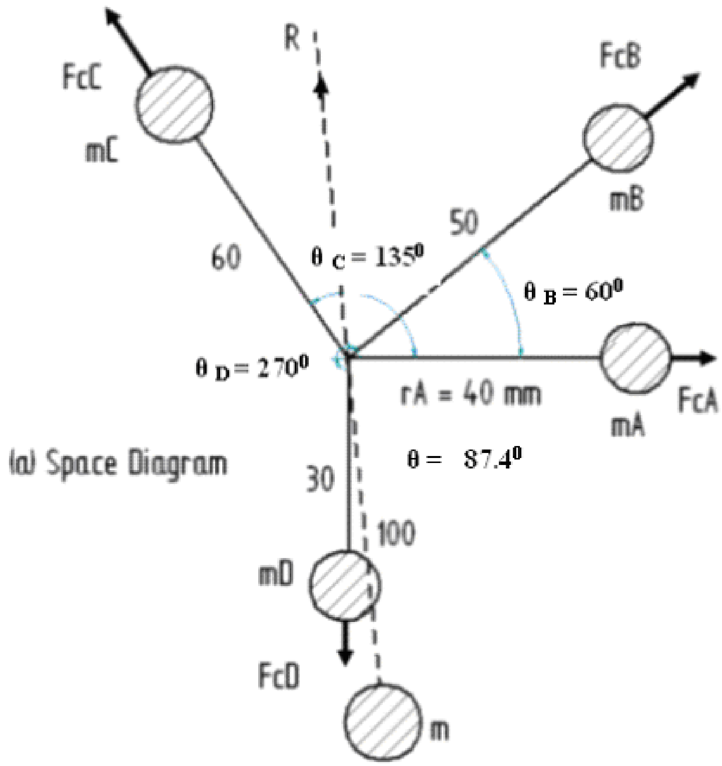
$$\text{and} \quad \theta = -87.4^0 \text{ or } 92.6^0$$

Remember ALL STUDENTS TAKE COPY i.e. in first quadrant all angles **($\sin\theta$, $\cos\theta$ and $\tan\theta$)** are positive, in second quadrant only **$\sin\theta$** is positive, in third quadrant only **$\tan\theta$** is positive and in fourth quadrant only **$\cos\theta$** is positive.

Since numerator is positive and denominator is negative, the resultant force makes with the horizontal, an angle (measured in the counter clockwise direction)

$$\theta = 92.6^0$$

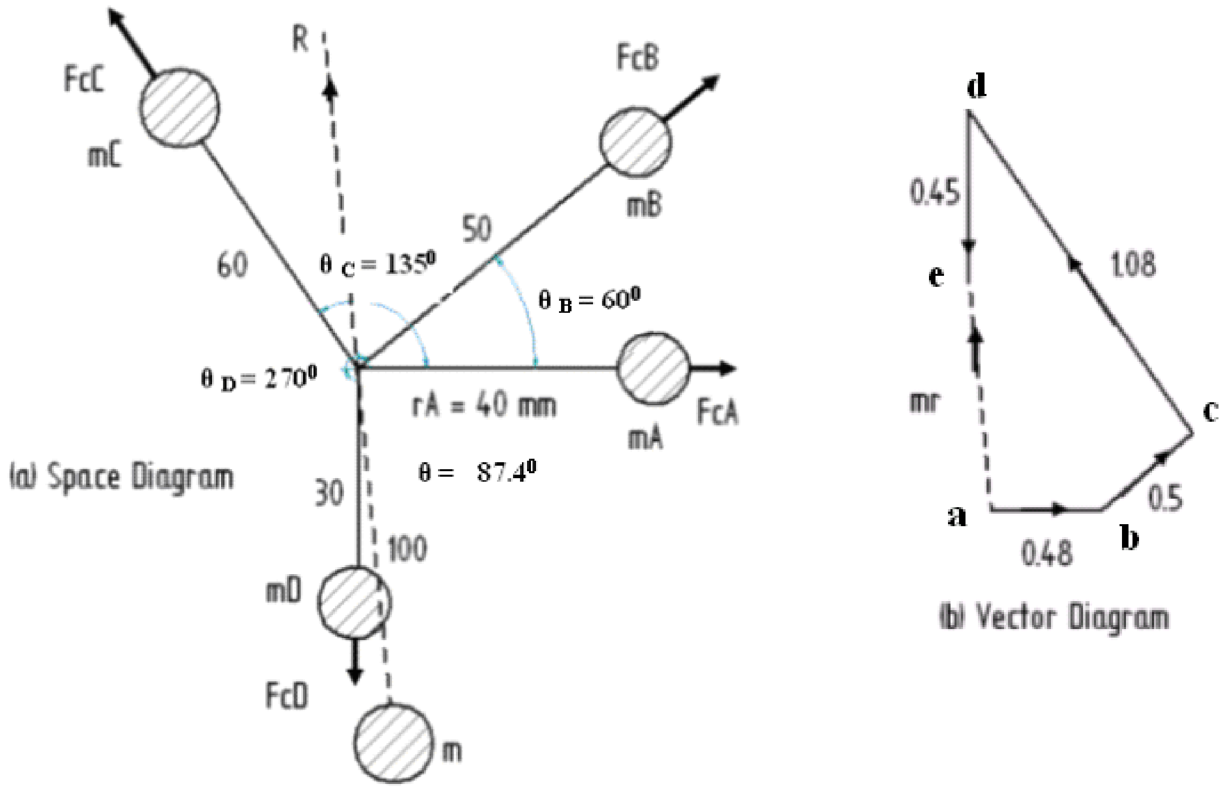
The balancing force is then equal to the resultant force, but in opposite direction.
The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta_M = 87.4^0$ angle measured in the clockwise direction.



Graphical Method:

Step 1:
Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

Draw the space diagram or angular position of the masses taking the actual angles(Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$).



Step 2:

Now draw the force polygon (The force polygon can be drawn by taking a convenient scale) by adding the known vectors as follows.

Draw a line 'ab' parallel to force F_{CA} (or the product $m_A r_A$ to a proper scale) of the space diagram. At 'b' draw a line 'bc' parallel to F_{CB} (or the product $m_B r_B$). Similarly draw lines 'cd', 'de' parallel to F_{CC} (or the product $m_C r_C$) and F_{CD} (or the product $m_D r_D$) respectively. The closing side 'ae' represents the resultant force 'R' in magnitude and direction as shown on the vector diagram.

Step 3:

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore,} \quad m = \frac{R}{r} = 7.48 \text{ kg Ans}$$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta_M = 87.4^0$ angle measured in the clockwise direction.

Problem 2:
The four masses A, B, C and D are 100 kg, 150 kg, 120 kg and 130 kg attached to a shaft and revolve in the same plane. The corresponding radii of rotations are 22.5 cm, 17.5 cm, 25 cm and 30 cm and the angles measured from A are 45^0, 120^0 and 255^0. Find the position and magnitude of the balancing mass, if the radius of rotation is 60 cm.

Solution:

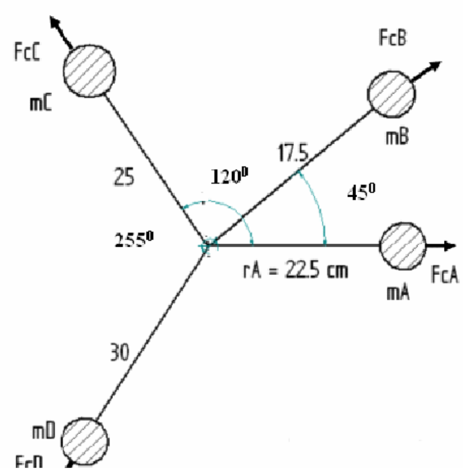
Analytical Method:

Given:

Mass(m) kg	Radius(r) m	Centrifugal force/ω^2 (m r) kg-m	Angle(θ)
m_A = 100 kg (reference mass)	r_A = 0.225 m	m_Ar_A = 22.5 kg-m	$\theta_A = 0^0$
m_B = 150 kg	r_B = 0.175 m	m_Br_B = 26.25 kg-m	$\theta_B = 45^0$
m_C = 120 kg	r_C = 0.250 m	m_Cr_C = 30 kg-m	$\theta_C = 120^0$
m_D = 130 kg	r_D = 0.300 m	m_Dr_D = 39 kg-m	$\theta_D = 255^0$
m =?	r = 0.60		θ = **?**

Step 1:
Draw the space diagram or angular position of the masses. Since all the angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$.

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.



Step 2:
Resolve the centrifugal forces horizontally and vertically and find their sum.

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ horizontally and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \cos\theta_i = m_A r_A \cos\theta_A + m_B r_B \cos\theta_B + m_C r_C \cos\theta_C + m_D r_D \cos\theta_D$$
$$= 22.5 \text{ x } \cos 0^0 + 26.25 \text{ x } \cos 45^0 + 30 \text{ x } \cos 120^0 + 39 \text{ x } \cos 255^0$$
$$= 22.5 + 18.56 + (-15) + (-10.1) = 15.97 \text{ kg} - \text{m} \quad ---------(1)$$

Resolving $m_A r_A$, $m_B r_B$, $m_C r_C$ and $m_D r_D$ vertically and taking their sum gives,

$$\sum_{i=1}^{n} m_i r_i \sin\theta_i = m_A r_A \sin\theta_A + m_B r_B \sin\theta_B + m_C r_C \sin\theta_C + m_D r_D \sin\theta_D$$
$$= 22.5 \text{ x } \sin 0^0 + 26.25 \text{ x } \sin 45^0 + 30 \text{ x } \sin 120^0 + 39 \text{ x } \sin 255^0$$
$$= 0 + 18.56 + 25.98 + (-37.67) = 6.87 \text{ kg} - \text{m} \quad ---------(2)$$

Step 3:
Determine the magnitude of the resultant centrifugal force

$$R = \sqrt{\left(\sum_{i=1}^{n} m_i r_i \cos\theta_i\right)^2 + \left(\sum_{i=1}^{n} m_i r_i \sin\theta_i\right)^2}$$
$$= \sqrt{(15.97)^2 + (6.87)^2} = 17.39 \text{ kg} - \text{m}$$

Step 4:
The balancing force is then equal to the resultant force, but in opposite direction. Now find out the magnitude of the balancing mass, such that

$$R = m\,r = 17.39 \text{ kg} - \text{m}$$
$$\text{Therefore,} \quad m = \frac{R}{r} = \frac{17.39}{0.60} = 28.98 \text{ kg Ans}$$

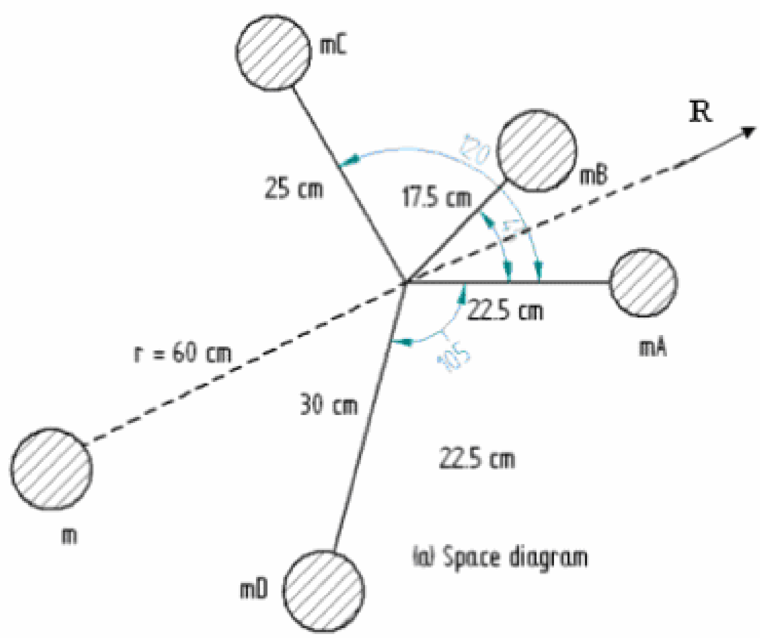
Where, m = balancing mass and r = its radius of rotation

Step 5:
Determine the position of the balancing mass 'm'.
If θ is the angle, which resultant force makes with the horizontal, then

$$\tan\theta = \frac{\sum_{i=1}^{n} m_i r_i \sin\theta_i}{\sum_{i=1}^{n} m_i r_i \cos\theta_i} = \frac{6.87}{15.97} = 0.4302$$

$$\text{and} \quad \theta = 23.28^0$$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203.28^0$ angle measured in the counter clockwise direction.



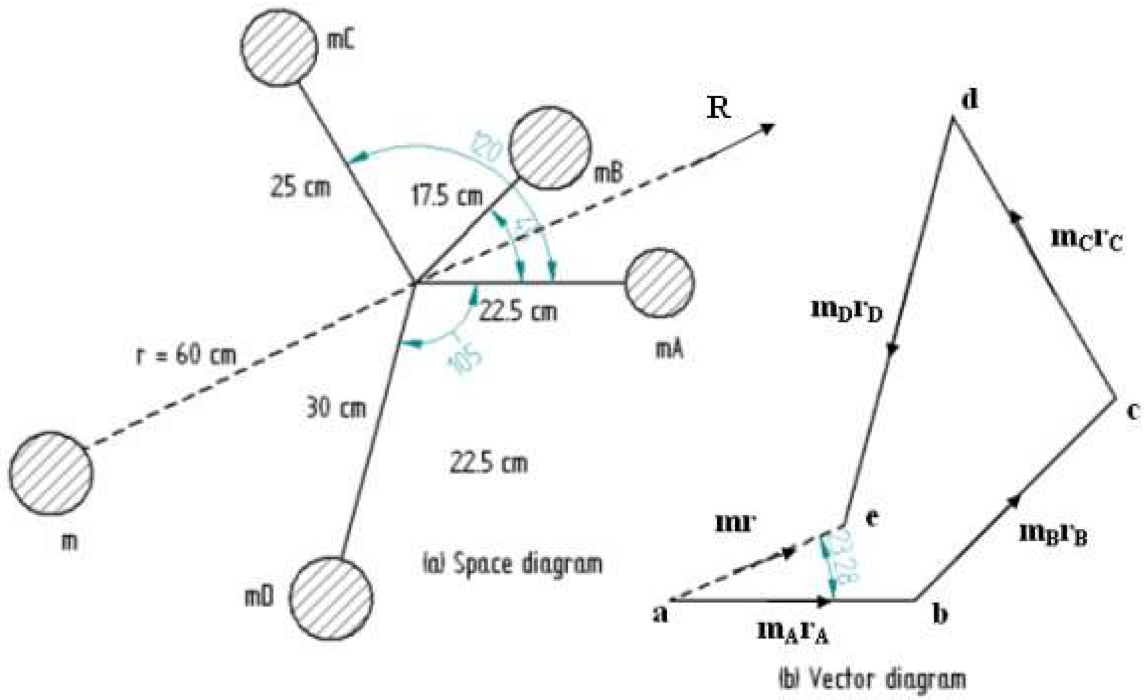
Graphical Method:

Step 1:

Tabulate the given data as shown. Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'mr' can be calculated and tabulated.

Step 2:

Draw the space diagram or angular position of the masses taking the actual angles (Since all angular position of the masses are given with respect to mass A, take the angular position of mass A as $\theta_A = 0^0$).



Draw a line 'ab' parallel to force F_{CA} (or the product $m_A r_A$ to a proper scale) of the space diagram. At 'b' draw a line 'bc' parallel to F_{CB} (or the product $m_B r_B$). Similarly draw lines 'cd', 'de' parallel to F_{CC} (or the product $m_C r_C$) and F_{CD} (or the product $m_D r_D$) respectively. The closing side 'ae' represents the resultant force 'R' in magnitude and direction as shown on the vector diagram.

Step 4:

The balancing force is then equal to the resultant force, but in opposite direction.

$$R = mr$$

$$\text{Therefore,} \quad m = \frac{R}{r} = 29 \text{ kg Ans}$$

The balancing mass 'm' lies opposite to the radial direction of the resultant force and the angle of inclination with the horizontal is, $\theta = 203^0$ angle measured in the counter clockwise direction.

Problem 3:
A rotor has the following properties.

Mass	magnitude	Radius	Angle	Axial distance from first mass
1	9 kg	100 mm	$\theta_A = 0^0$	-
2	7 kg	120 mm	$\theta_B = 60^0$	160 mm
3	8 kg	140 mm	$\theta_C = 135^0$	320 mm
4	6 kg	120 mm	$\theta_D = 270^0$	560 mm

If the shaft is balanced by two counter masses located at 100 mm radii and revolving in planes midway of planes 1 and 2, and midway of 3 and 4, determine the magnitude of the masses and their respective angular positions.

Solution:

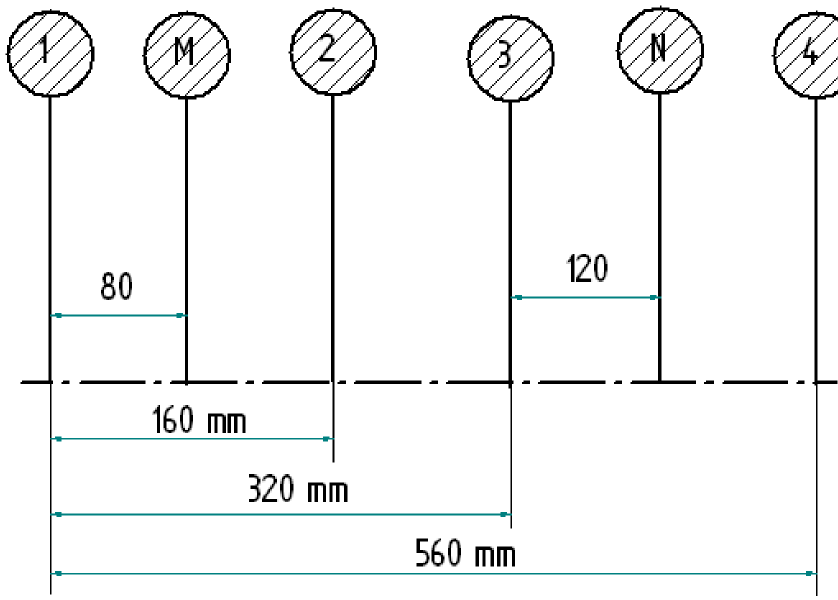
Analytical Method:

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'M' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
1	9.0	0.10	$m_1 r_1 = 0.9$	-0.08	-0.072	0^0
M	m_M = ?	0.10	$m_M r_M = 0.1 m_M$	0	0	$\theta_M = ?$
2	7.0	0.12	$m_2 r_2 = 0.84$	0.08	0.0672	60^0
3	8.0	0.14	$m_3 r_3 = 1.12$	0.24	0.2688	135^0
N	m_N = ?	0.10	$m_N r_N = 0.1 m_N$	0.36	$m_N r_N l_N = 0.036 m_N$	$\theta_N = ?$
4	6.0	0.12	$m_4 r_4 = 0.72$	0.48	0.3456	270^0

For dynamic balancing the conditions required are,

$$\sum mr + m_M r_M + m_N r_N = 0 \text{ ----------(I)} \quad \text{for force balance}$$

$$\sum mrl + m_N r_N l_N = 0 \text{ --------------(II)} \quad \text{for couple balance}$$



(a) Position of planes of masses

Step 1:
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_N r_N l_N \cos\theta_N = 0$$

On substitution we get

$$-0.072\cos 0^0 + 0.0672\cos 60^0 + 0.2688\cos 135^0$$

$$+0.3456\cos 270^0 + 0.036\, m_N \cos\theta_N = 0$$

$$\text{i.e. } 0.036\, m_N \cos\theta_N = 0.2285 - - - - - (1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_N r_N l_N \sin\theta_N = 0$$

On substitution we get

$$-0.072\sin 0^0 + 0.0672\sin 60^0 + 0.2688\sin 135^0$$

$$+0.3456\sin 270^0 + 0.036\, m_N \sin\theta_N = 0$$

$$\text{i.e. } 0.036\, m_N \sin\theta_N = 0.09733 - - - - - (2)$$

Squaring and adding (1) and (2), we get

$$m_N r_N l_N = \sqrt{(0.2285)^2 + (0.09733)^2}$$
$$\text{i.e., } 0.036 m_N = 0.2484$$
$$\text{Therefore, } \quad m_N = \frac{0.2484}{0.036} = 6.9\text{kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_N = \frac{0.09733}{0.2285} \quad \text{and } \theta_N = 23.07^0$$

Step 2:
Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_M r_M \cos\theta_M + m_N r_N \cos\theta_N = 0$$
On substitution we get
$$0.9\cos 0^0 + 0.84\cos 60^0 + 1.12\cos 135^0 + 0.72\cos 270^0$$
$$+ m_M r_M \cos\theta_M + 0.1\text{x}6.9\text{xcos}23.07^0 = 0$$
$$\text{i.e. } m_M r_M \cos\theta_M = -1.1629 -----(3)$$

Sum of the vertical components gives,

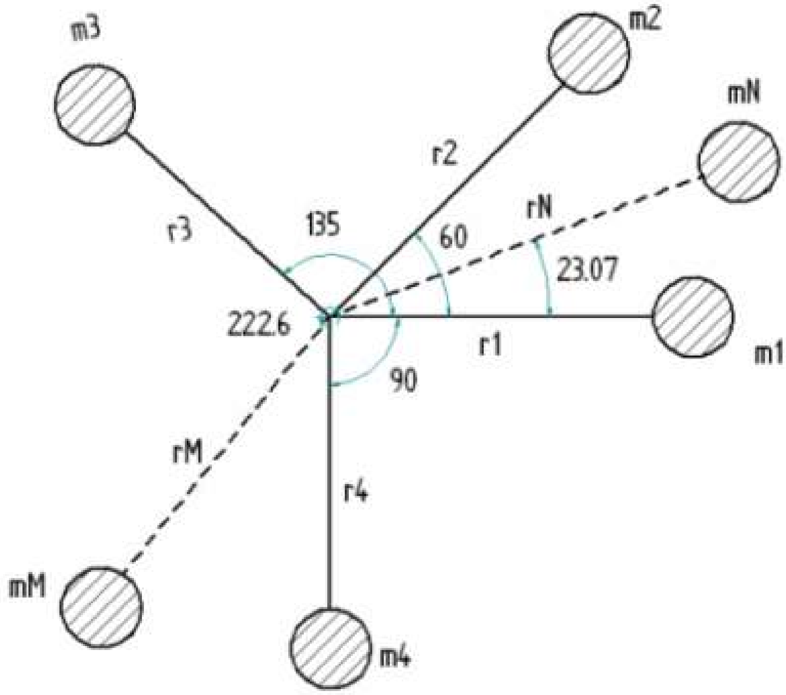
$$\sum mr\sin\theta + m_M r_M \sin\theta_M + m_N r_N \sin\theta_N = 0$$
On substitution we get
$$0.9\sin 0^0 + 0.84\sin 60^0 + 1.12\sin 135^0 + 0.72\sin 270^0$$
$$+ m_M r_M \sin\theta_M + 0.1\text{x}6.9\text{xsin}23.07^0 = 0$$
$$\text{i.e. } m_M r_M \sin\theta_M = -1.0698 -----(4)$$

Squaring and adding (3) and (4), we get

$$m_M r_M = \sqrt{(-1.1629)^2 + (-1.0698)^2}$$
$$\text{i.e., } 0.1 m_M = 1.580$$
$$\text{Therefore, } \quad m_M = \frac{1.580}{0.1} = 15.8\text{ kg Ans}$$

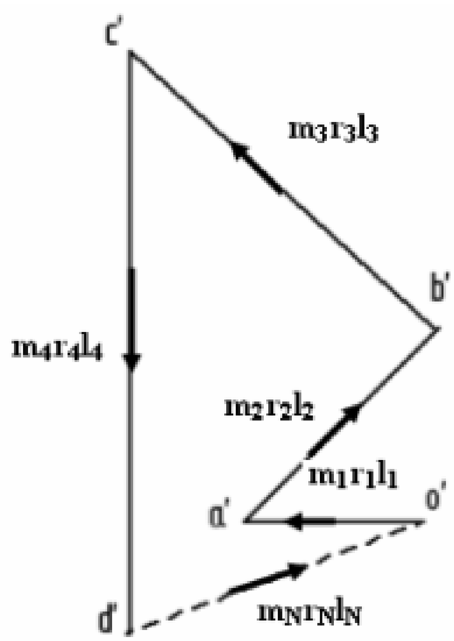
Dividing (4) by (3), we get

$$\tan\theta_M = \frac{-1.0698}{-1.1629} \quad \text{and } \theta_M = 222.61^0 \text{ Ans}$$

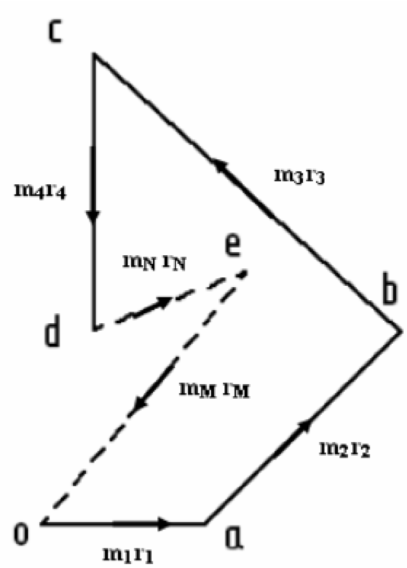


(b) Angular position of masses

Graphical Solution:



(c) Couple polygon



(d) Force polygon

Problem 4:
The system has the following data.

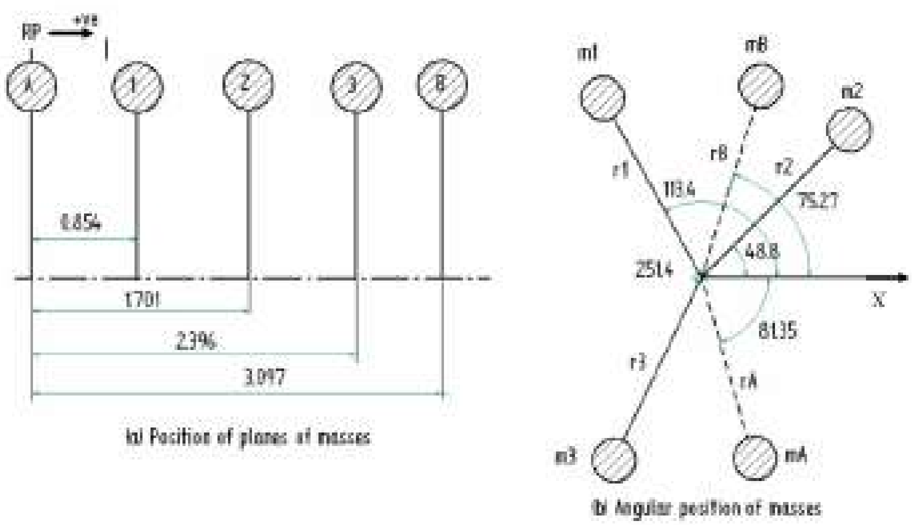
$\mathbf{m_1} = 1.2\ \mathbf{kg}$	$\mathbf{r_1} = 1.135\ \mathbf{m} @ \angle 113.4^0$
$\mathbf{m_1} = 1.8\ \mathbf{kg}$	$\mathbf{r_2} = 0.822\ \mathbf{m} @ \angle 48.8^0$
$\mathbf{m_1} = 2.4\ \mathbf{kg}$	$\mathbf{r_3} = 1.04\ \mathbf{m} @ \angle 251.4^0$

The distances of planes in metres from plane A are:

$$l_1 = 0.854\ ,\ l_2 = 1.701\ ,l_3 = 2.396\ ,l_B = 3.097$$

Find the mass-radius products and their angular locations needed to dynamically balance the system using the correction planes A and B.

Solution: **Analytical Method**



(a) Position of planes of masses

(b) Angular position of masses

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'A' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	m_A	r_A	$m_A\ r_A$ =?	0	0	$\theta_A = ?$
1	1.2	1.135	1.362	0.854	1.163148	113.4^0
2	1.8	0.822	1.4796	1.701	2.5168	48.8^0
3	2.4	1.04	2.496	2.396	5.9804	251.4^0
B	m_B	r_B	$m_B\ r_B$ =?	3.097	3.097 $m_B\ r_B$	$\theta_B = ?$

Step 1:

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_B r_B l_B \cos\theta_B = 0$$

On substitution we get

$$1.163148\cos 113.4^0 + 2.5168\cos 48.8^0 + 5.9804\cos 251.4^0$$
$$+3.097\, m_B r_B \cos\theta_B = 0$$
$$\text{i.e. } m_B r_B \cos\theta_B = \frac{0.71166}{3.097} - - - - - (1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_B r_B l_B \sin\theta_B = 0$$

On substitution we get

$$1.163148\sin 113.4^0 + 2.5168\sin 48.8^0 + 5.9804\sin 251.4^0$$
$$+3.097\, m_B r_B \sin\theta_B = 0$$
$$\text{i.e. } m_B r_B \sin\theta_B = \frac{2.7069}{3.097} - - - - - (2)$$

Squaring and adding (1) and (2), we get

$$m_B r_B = \sqrt{\left(\frac{0.71166}{3.097}\right)^2 + \left(\frac{2.7069}{3.097}\right)^2}$$
$$= 0.9037\ \text{kg} - \text{m}$$

Dividing (2) by (1), we get

$$\tan\theta_B = \frac{2.7069}{0.71166} \quad \text{and} \quad \theta_B = 75.27^0 \ \text{Ans}$$

Step 2:

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_A r_A \cos\theta_A + m_B r_B \cos\theta_B = 0$$

On substitution we get

$$1.362\cos 113.4^0 + 1.4796\cos 48.8^0 + 2.496\cos 251.4^0$$
$$+ m_A r_A \cos\theta_A + 0.9037\cos 75.27^0 = 0$$

Therefore

$$m_A r_A \cos\theta_A = 0.13266 ---------- (3)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_A r_A \sin\theta_A + m_B r_B \sin\theta_B = 0$$

On substitution we get

$$1.362\sin 113.4^0 + 1.4796\sin 48.8^0 + 2.496\sin 251.4^0$$
$$+ m_A r_A \sin\theta_A + 0.9037\sin 75.27^0 = 0$$

Therefore

$$m_A r_A \sin\theta_A = -0.87162 ---------- (4)$$

Squaring and adding (3) and (4), we get

$$m_A r_A = \sqrt{(0.13266)^2 + (-0.87162)^2}$$
$$= 0.8817 \text{ kg} - \text{m}$$

Dividing (4) by (3), we get

$$\tan\theta_A = \frac{-0.87162}{0.13266} \quad \text{and} \quad \theta_A = -81.35^0 \text{ Ans}$$

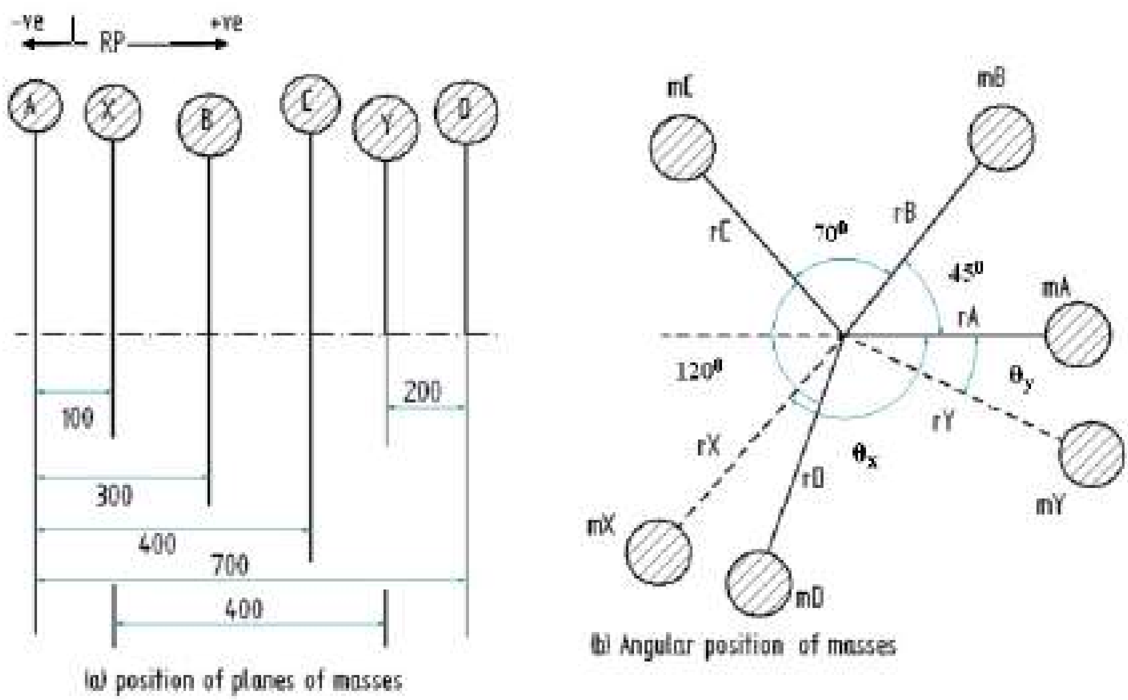
Problem 5:

A shaft carries four masses A, B, C and D of magnitude 200 kg, 300 kg, 400 kg and 200 kg respectively and revolving at radii 80 mm, 70 mm, 60 mm and 80 mm in planes measured from A at 300 mm, 400 mm and 700 mm. The angles between the cranks measured anticlockwise are A to B 45^0, B to C 70^0 and C to D 120^0. The balancing masses are to be placed in planes X and Y. The distance between the planes A and X is 100 mm, between X and Y is 400 mm and between Y and D is 200 mm. If the balancing masses revolve at a radius of 100 mm, find their magnitudes and angular positions.

Graphical solution:

Let, m_X be the balancing mass placed in plane X and m_Y be the balancing mass placed in plane Y which are to be determined.

Step 1:
Draw the position of the planes as shown in figure (a).



(a) position of planes of masses

(b) Angular position of masses

Let X be the reference plane (R.P.). The distances of the planes to the right of the plane X are taken as positive (+ve) and the distances of planes to the left of X plane are taken as negative(-ve). The data may be tabulated as shown

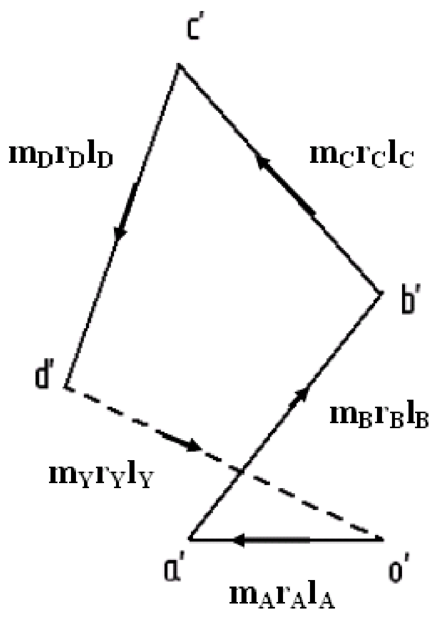
Since the magnitude of the centrifugal forces are proportional to the product of the mass and its radius, the product 'm r' can be calculated and tabulated. Similarly the magnitude of the couples are proportional to the product of the mass , its radius and the axial distance from the reference plane, the product 'm r l' can be calculated and tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'X' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	200	0.08	$m_A r_A = 16$	-0.10	-1.60	-
X	m_X =?	0.10	$m_X r_X = 0.1 m_X$	0	0	θ_x =?
B	300	0.07	$m_B r_B = 21$	0.20	4.20	A to B 45^0
C	400	0.06	$m_C r_C = 24$	0.30	7.20	B to C 70^0
Y	m_Y =?	0.10	$m_Y r_Y = 0.1 m_Y$	0.40	$m_Y r_Y l_Y = 0.04 m_Y$	θ_Y =?
D	200	0.08	$m_D r_D = 16$	0.60	9.60	C to D 120^0

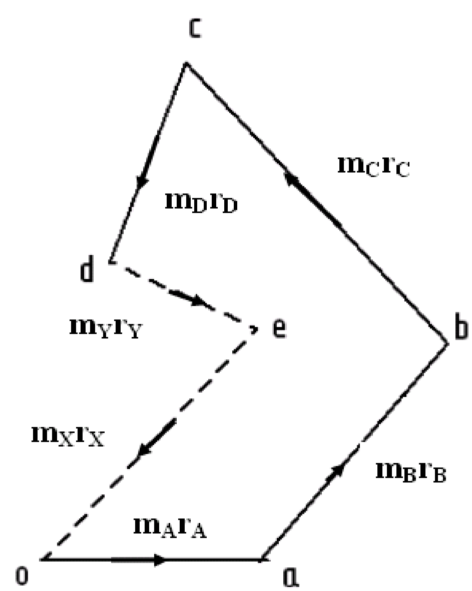
Step 2:

Assuming the mass A as horizontal draw the sketch of angular position of masses as shown in figure (b).

Step 3:
Draw the couple polygon to some suitable scale by taking the values of 'm r l' (column no. 6) of the table as shown in figure (c).



(c) Couple polygon



(d) Force polygon

Draw line o'a' parallel to the radial line of mass m_A.
At a' draw line a'b' parallel to radial line of mass m_B.
Similarly, draw lines b'c', c'd' parallel to radial lines of masses m_C and m_D respectively.
Now, join d' to o' which gives the balanced couple.

We get, $$0.04\ m_Y = \text{vector } d'o' = 7.3\ kg - m^2$$
$$\text{or} \quad m_Y = 182.5\ kg \ \text{Ans}$$

Step 4:
To find the angular position of the mass m_Y draw a line om_Y in figure (b) parallel to d'o' of the couple polygon.

By measurement we get $\theta_Y = 12^0$ in the clockwise direction from m_A.

Step 5:
Now draw the force polygon by considering the values of 'm r' (column no. 4) of the table as shown in figure (d).
Follow the similar procedure of step 3. The closing side of the force polygon i.e. 'e o' represents the balanced force.

$$m_X r_X = \text{vector } eo = 35.5\ kg - m$$
$$\text{or} \quad m_X = 355\ kg \ \text{Ans}$$

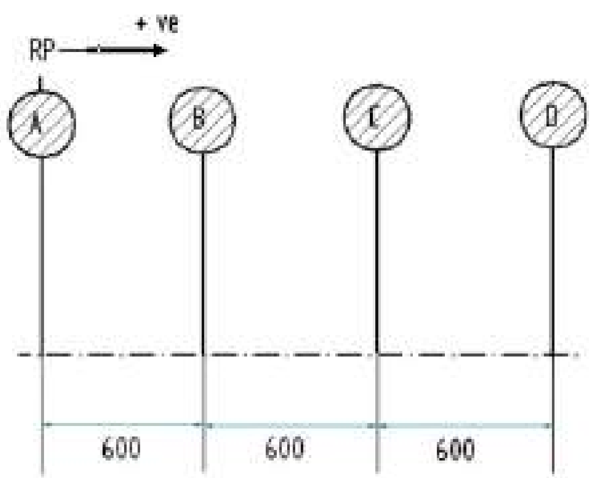
Step 6:
The angular position of m_X is determined by drawing a line om_X parallel to the line 'e o' of the force polygon in figure (b). From figure (b) we get,
$\theta_X = 145^0$, measured clockwise from m_A. Ans

Problem 6:
A, B, C and D are four masses carried by a rotating shaft at radii 100 mm, 125 mm, 200 mm and 150 mm respectively. The planes in which the masses revolve are spaced 600 mm apart and the mass of B, C and D are 10 kg, 5 kg and 4 kg respectively. Find the required mass A and relative angular settings of the four masses so that the shaft shall be in complete balance.
Solution:

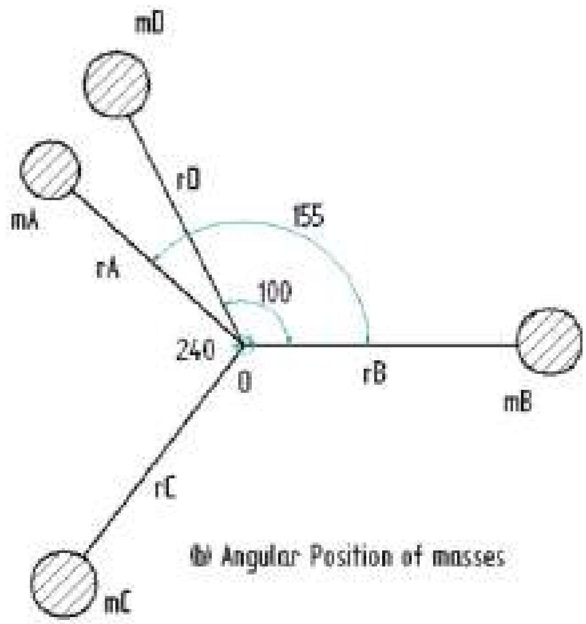
Graphical Method:

Step 1:
Let, m_A be the balancing mass placed in plane A which is to be determined along with the relative angular settings of the four masses.
Let A be the reference plane (R.P.).
Assume the mass B as horizontal
Draw the sketch of angular position of mass m_B (line om_B) as shown in figure (b). The data may be tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'A' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	m_A= ?	0.1	$m_A r_A = 0.1 m_A$	0	0	θ_A =?
B	10	0.125	$m_B r_B = 1.25$	0.6	0.75	θ_B =0
C	5	0.2	$m_C r_C = 1.0$	1.2	1.2	θ_C =?
D	4	0.15	$m_D r_D = 0.6$	1.8	1.08	θ_D =?



(a) Position of planes of masses

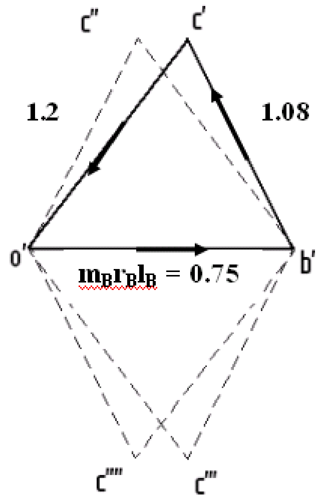


(b) Angular Position of masses

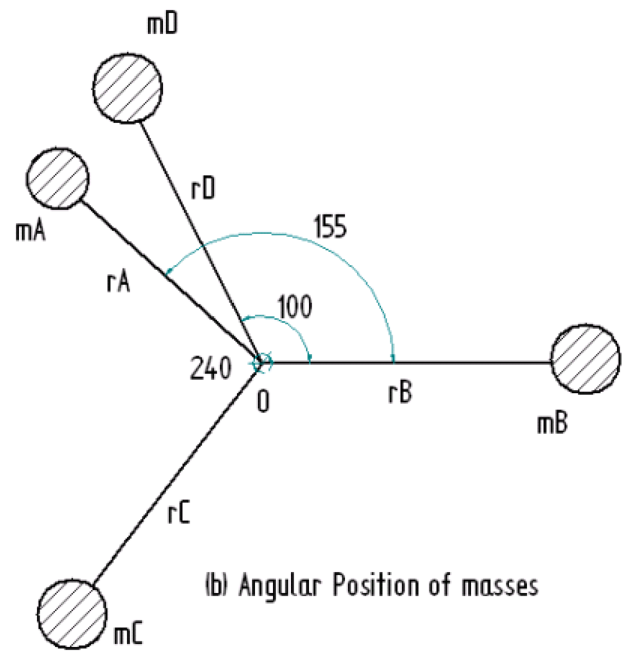
Draw a line o'b' equal to 0.75 kg-m^2 parallel to the line om_B. At point o' and b' draw vectors o'c' and b'c' equal to 1.2 kg-m^2 and 1.08 kg-m^2 respectively. These vectors intersect at point c'.

For the construction of force polygon there are four options.

Any one option can be used and relative to that the angular settings of mass C and D are determined.



(c) Couple polygon

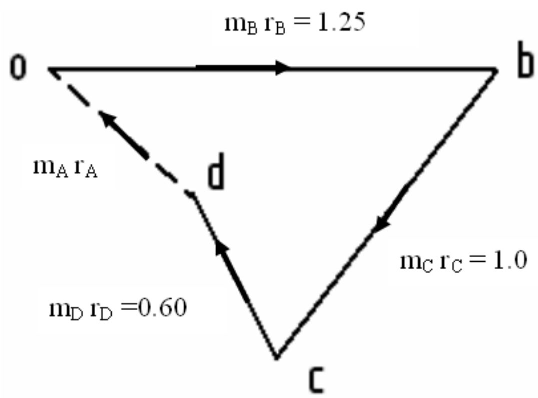


(b) Angular Position of masses

$$\theta_D = 100^0 \quad \text{and} \quad \theta_C = 240^0 \quad \text{Ans}$$

Step 4:

In order to find m_A and its angular setting draw the force polygon as shown in figure (d).



(d) Force polygon

Closing side of the force polygon od represents the product $m_A\, r_A$. i.e.

$$m_A\ r_A = 0.70\ kg\text{-}m$$

$$\text{Therefore,} \quad m_A = \frac{0.70}{r_A} = 7\ kg \quad \text{Ans}$$

Step 5:

Now draw line om_A parallel to od of the force polygon. By measurement, we get,

$$\theta_A = 155^0 \quad \text{Ans}$$

Problem 7:
A shaft carries three masses A, B and C. Planes B and C are 60 cm and 120 cm from A. A , B and C are 50 kg, 40 kg and 60 kg respectively at a radius of 2.5 cm. The angular position of mass B and mass C with A are 90^0 and 210^0 respectively. Find the unbalanced force and couple. Also find the position and magnitude of balancing mass required at 10 cm radius in planes L and M midway between A and B, and B and C.

Solution:

Case (i):

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'A' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	50	0.025	$m_A\ r_A$ = 1.25	0	0	$\theta_A = 0^0$
B	40	0.025	$m_B\ r_B$ = 1.00	0.6	0.6	$\theta_B = 90^0$
C	60	0.025	$m_C\ r_C$ = 1.50	1.2	1.8	$\theta_C = 210^0$

Analytical Method

Step 1:
Determination of unbalanced couple
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum \mathbf{mrl\ cos\theta = 0.6\ cos\ 90^0 + 1.8\ cos\ 210^0 = -1.559 - - - - - (1)}$$

Sum of the vertical components gives,

$$\sum \mathbf{mrl\ sin\theta = 0.6\ sin\ 90^0 + 1.8\ sin\ 210^0 = -0.3 - - - - - (2)}$$

Squaring and adding (1) and (2), we get

$$\mathbf{C_{unbalanced}} = \sqrt{(\mathbf{-1.559})^2 + (\mathbf{-0.3})^2}$$
$$= \mathbf{1.588\ kg - m^2}$$

Step 2:
Determination of unbalanced force

Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum \mathbf{mr\ cos\theta} = \mathbf{1.25\ cos\ 0^0 + 1.0\ cos\ 90^0 + 1.5\ cos 210^0}$$
$$= \mathbf{1.25 + 0 + (-1.299) = -0.049 ---------(3)}$$

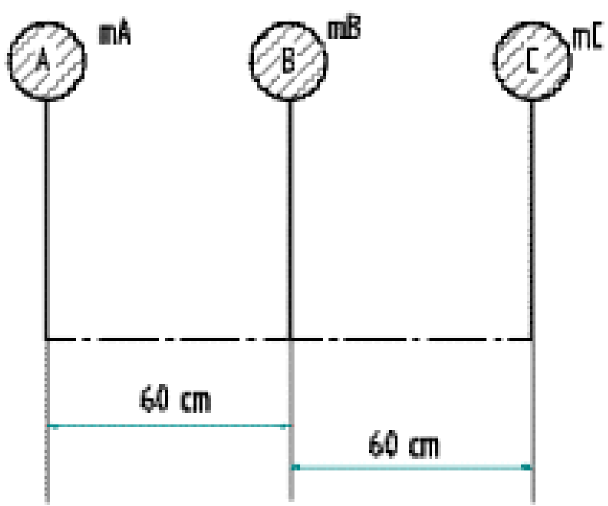
Sum of the vertical components gives,

$$\sum \mathbf{mr\ sin\theta} = \mathbf{1.25\ sin\ 0^0 + 1.0\ sin\ 90^0 + 1.5\ sin\ 210^0}$$
$$= \mathbf{0 + 1.0 + (-0.75) = 0.25 ---------(4)}$$

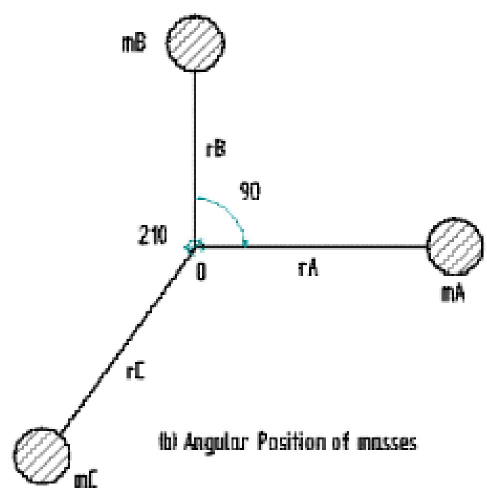
Squaring and adding (3) and (4), we get

$$\mathbf{F_{unbalanced}} = \sqrt{(\mathbf{-0.049})^2 + (\mathbf{0.25})^2}$$
$$= \mathbf{0.2548\ \ kg - m}$$

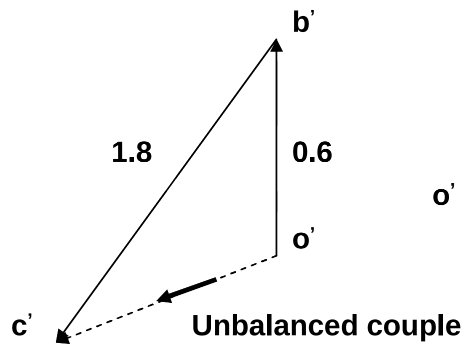
Graphical solution:



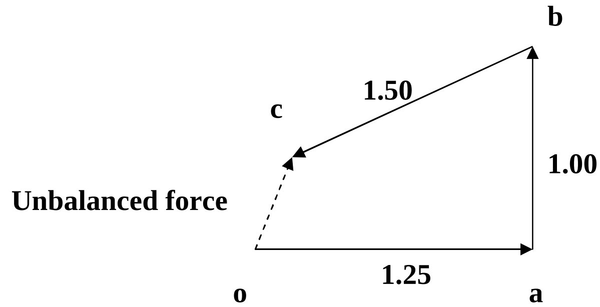
(a) Position of planes of masses



(b) Angular Position of masses

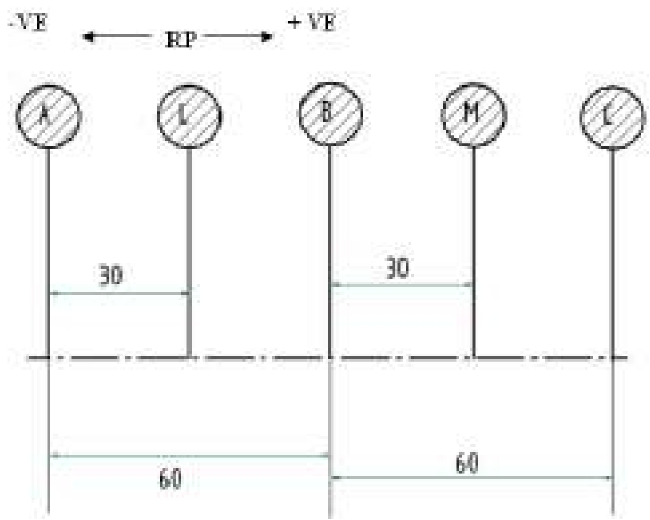


Couple polygon

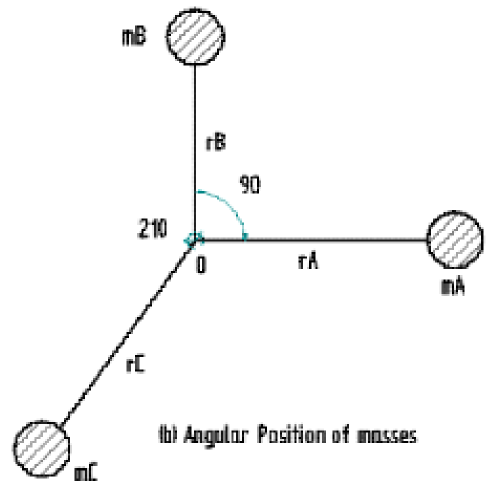


Force polygon

Case (ii):



(a) Position of planes of masses



(b) Angular Position of masses

To determine the magnitude and directions of masses m_M and m_L.

Let, m_L be the balancing mass placed in plane L and m_M be the balancing mass placed in plane M which are to be determined.

The data may be tabulated as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'L' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A	50	0.025	$m_A\, r_A = 1.25$	-0.3	-0.375	$\theta_A = 0^0$
L (R.P.)	m_L = ?	0.10	0.1 m_L	0	0	θ_L =?
B	40	0.025	$m_B\, r_B = 1.00$	0.3	0.3	$\theta_B = 90^0$
M	m_M = ?	0.10	0.1 m_M	0.6	0.06 m_M	θ_M =?
C	60	0.025	$m_C\, r_C = 1.50$	0.9	1.35	$\theta_C = 210^0$

Analytical Method:

Step 1:
Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mrl\cos\theta + m_M r_M l_M \cos\theta_M = 0$$

On substitution we get

$$-0.375\cos 0^0 + 0.3\cos 90^0 + 0.06\, m_M\cos\theta_M + 1.35\cos 210^0 = 0$$

$$\text{i.e. } -0.375 + 0 + 0.06\, m_M\cos\theta_M + (-1.16913) = 0$$

$$0.06\, m_M\cos\theta_M = 1.54413$$

$$m_M\cos\theta_M = \frac{1.54413}{0.06} = 25.74 \quad -----(1)$$

Sum of the vertical components gives,

$$\sum mrl\sin\theta + m_M r_M l_M \sin\theta_M = 0$$

On substitution we get

$$-0.375\sin 0^0 + 0.3\sin 90^0 + 0.06\, m_M\sin\theta_M + 1.35\sin 210^0 = 0$$

$$\text{i.e. } 0 + 0.3 + 0.06\, m_M\sin\theta_M + (-0.675) = 0$$

$$0.06\, m_M\sin\theta_M = 0.375$$

$$m_M\sin\theta_M = \frac{0.375}{0.06} = 6.25 \quad -----(2)$$

Squaring and adding (1) and (2), we get

$$(m_M \cos\theta_M)^2 + (m_M \sin\theta_M)^2 = (25.74)^2 + (6.25)^2 = 701.61$$

$$\text{i.e. } m_M^2 = 701.61 \qquad \text{and} \quad m_M = 26.5 \text{ kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_M = \frac{6.25}{25.74} \quad \text{and} \;\; \theta_M = 13.65^0 \text{ Ans}$$

Step 2:
Resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\cos\theta + m_L r_L \cos\theta_L + m_M r_M \cos\theta_M = 0$$

On substitution we get

$$1.25\cos 0^0 + 0.1m_L \cos\theta_L + 1.0\cos 90^0 + 2.649\cos 13.65^0 + 1.5\cos 210^0 = 0$$

$$1.25 + 0.1m_L \cos\theta_L + 0 + 2.5741 + (-1.299) = 0$$

Therefore

$$0.1m_L \cos\theta_L + 2.5251 = 0$$

$$\text{and} \qquad m_L \cos\theta_L = \frac{-2.5251}{0.1} = -25.251 \text{---------}(3)$$

Sum of the vertical components gives,

$$\sum mr\sin\theta + m_L r_L \sin\theta_L + m_M r_M \sin\theta_M = 0$$

On substitution we get

$$1.25\sin 0^0 + 0.1m_L \sin\theta_L + 1.0\sin 90^0 + 2.649\sin 13.65^0 + 1.5\sin 210^0 = 0$$

$$0 + 0.1m_L \sin\theta_L + 1 + 0.6251 + (-0.75) = 0$$

Therefore

$$0.1m_L \sin\theta_L + 0.8751 = 0$$

$$\text{and} \qquad m_L \sin\theta_L = \frac{-0.8751}{0.1} = -8.751 \text{---------}(4)$$

Squaring and adding (3) and (4), we get

$$(m_L \cos\theta_L)^2 + (m_L \sin\theta_L)^2 = (-25.251)^2 + (-8.751)^2 = 714.193$$

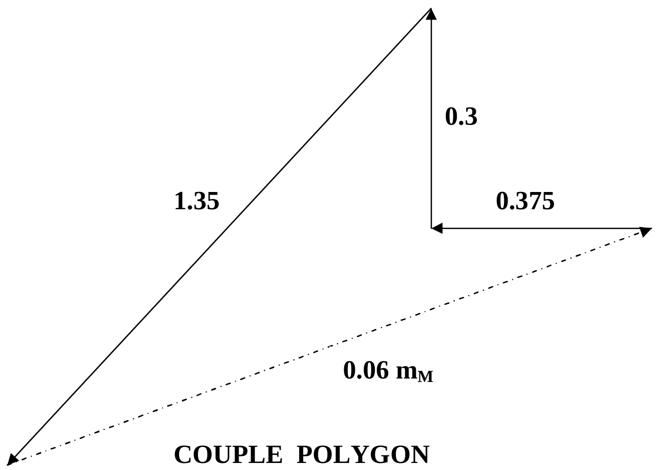
$$\text{i.e. } m_L^2 = 714.193 \qquad \text{and} \quad m_L = 26.72 \text{ kg Ans}$$

Dividing (4) by (3), we get

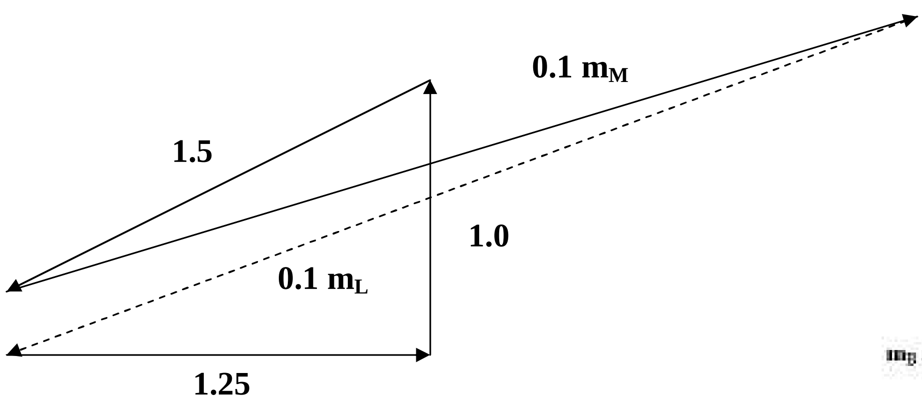
$$\tan\theta_L = \frac{-8.751}{-25.251} \quad \text{and} \quad \theta_L = 19.11^0 \text{ Ans}$$

The balancing mass m_L is at an angle $19.11^0 + 180^0 = 199.11^0$ measured in counter clockwise direction.

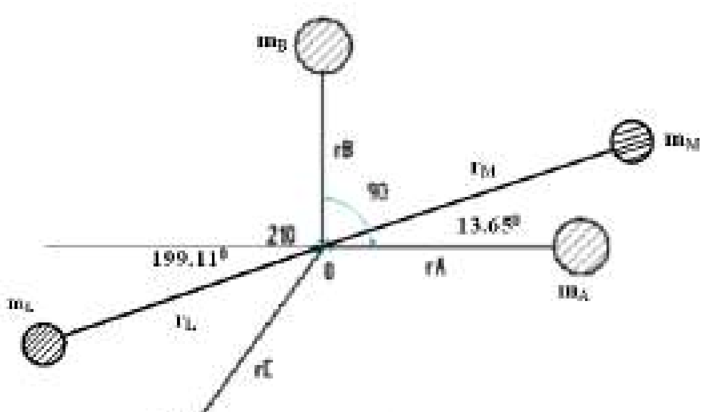
Graphical Method:



COUPLE POLYGON



FORCE POLYGON



(b) Angular Position of masses

Problem 8:

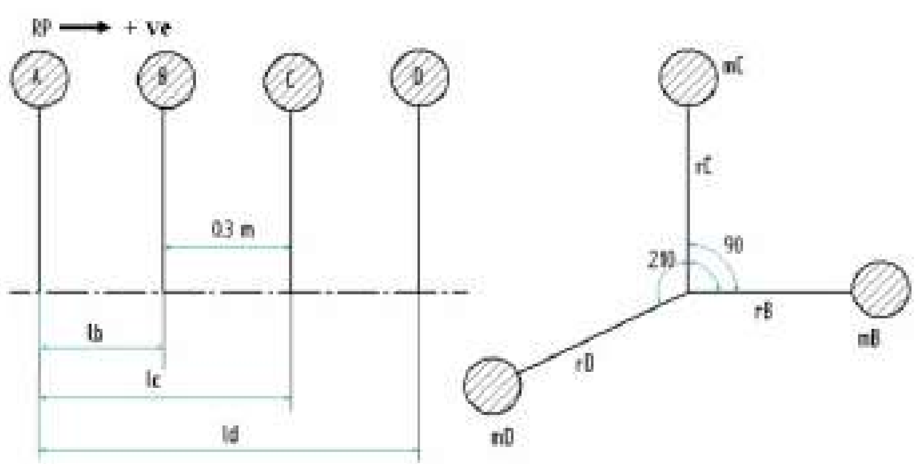
Four masses A, B, C and D are completely balanced. Masses C and D make angles of 90^0 and 210^0 respectively with B in the same sense. The planes containing B and C are 300 mm apart. Masses A, B, C and D can be assumed to be concentrated at radii of 360 mm, 480 mm, 240 mm and 300 mm respectively. The masses B, C and D are 15 kg, 25 kg and 20 kg respectively. Determine i) mass A and its angular position ii) position of planes A and D.

Solution: **Analytical Method**

Step 1:
Draw the space diagram or angular position of the masses. Since the angular position of the masses C and D are given with respect to mass B, take the angular position of mass B as $\theta_B = 0^0$.

Tabulate the given data as shown.

Plane 1	Mass (m) kg 2	Radius (r) m 3	Centrifugal force/ω^2 (m r) kg-m 4	Distance from Ref. plane 'A' m 5	Couple/ ω^2 (m r L) kg-m^2 6	Angle θ 7
A (R.P.)	m_A= ?	0.36	$m_A\ r_A$ = 0.36 m_A	0	0	θ_A =**?**
B	15	0.48	$m_B\ r_B$ = 7.2	l_B = ?	7.2 l_B	$\theta_B = 0$
C	25	0.24	$m_C\ r_C$ = 6.0	l_C = ?	6.0 l_C	$\theta_C = 90^0$
D	20	0.30	$m_D\ r_D$ = 6.0	l_D = ?	6.0 l_D	$\theta_D = 210^0$



(a) Position of planes of masses (Assumed)   (b) Angular position of masses

Step 2:
Mass m_A be the balancing mass placed in plane A which is to be determined along with its angular position.

Refer column 4 of the table. Since m_A is to be determined (which is the only unknown) ,resolve the forces into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum mr\ \cos\theta = m_A\, r_A\ \cos\theta_A + m_B\, r_B\ \cos\theta_B + m_C\, r_C\ \cos\theta_C + m_D\, r_D\ \cos\theta_D = 0$$

On substitution we get

$$0.36\, m_A \cos\theta_A + 7.2\ \cos 0^0 + 6.0\ \cos 90^0 + 6.0\ \cos 210^0 = 0$$

Therefore

$$0.36\ m_A \cos\theta_A = -2.004 ---------- (1)$$

Sum of the vertical components gives,

$$\sum mr\ \sin\theta = m_A\, r_A\ \sin\theta_A + m_B\, r_B\ \sin\theta_B + m_C\, r_C\ \sin\theta_C + m_D\, r_D\ \sin\theta_D = 0$$

On substitution we get

$$0.36\ m_A \sin\theta_A + 7.2\ \sin 0^0 + 6.0\ \sin 90^0 + 6.0\ \sin 210^0 = 0$$

Therefore

$$0.36\ m_A \sin\theta_A = -3.0 ---------- (2)$$

Squaring and adding (1) and (2), we get

$$0.36^2\,(m_A)^2 = (-2.004)^2 + (-3.0)^2 = 13.016$$

$$m_A = \sqrt{\frac{13.016}{0.36^2}} = 10.02\ \text{kg Ans}$$

Dividing (2) by (1), we get

$$\tan\theta_A = \frac{-3.0}{-2.004}\ \text{ and Resutltant makes an angle} = 56.26^0$$

$$\text{The balancing mass A makes an angle of } \theta_A = 236.26^0\ \text{Ans}$$

Step 3:

Resolve the couples into their horizontal and vertical components and find their sums.

Sum of the horizontal components gives,

$$\sum \mathbf{mr\,l\,\cos\theta = m_A\,r_A\,l_A\cos\theta_A + m_B\,r_B\,l_B\cos\theta_B + m_C\,r_C\,l_C\cos\theta_C + m_D\,r_D\,l_D\cos\theta_D = 0}$$

On substitution we get

$$\mathbf{0 + 7.2l_B\cos 0^0 + 6.0l_C\cos 90^0 + 6.0l_D\cos 210^0 = 0}$$

$$\mathbf{7.2l_B - 5.1962l_D = 0 \text{-----------} (3)}$$

Sum of the vertical components gives,

$$\sum \mathbf{mr\,l}\ \sin\mathbf{\theta = m_A\,r_A\,l_A}\sin\mathbf{\theta_A + m_B\,r_B\,l_B}\sin\mathbf{\theta_B + m_C\,r_C\,l_C}\sin\mathbf{\theta_C + m_D\,r_D\,l_D}\sin\mathbf{\theta_D = 0}$$

On substitution we get

$$\mathbf{0 + 7.2l_B}\sin\mathbf{0^0 + 6.0l_C}\sin\mathbf{90^0 + 6.0l_D}\sin\mathbf{210^0 = 0}$$

$$\mathbf{0} + 0 + \mathbf{6.0l_C} - 3\mathbf{l_D} = \mathbf{0 \text{-----------} (4)}$$

But from figure we have, $\mathbf{l_C} = \mathbf{l_B} + 0.3$

On substituting this in equation (4), we get

$$\mathbf{6.0}(\mathbf{l_B} + 0.3) - 3\mathbf{l_D} = \mathbf{0}$$

$$\textbf{i.e. } \mathbf{6.0l_B} - 3\mathbf{l_D} = \mathbf{1.8 \text{----------} (5)}$$

Thus we have two equations (3) and (5), and two unknowns $\mathbf{l_B, l_D}$

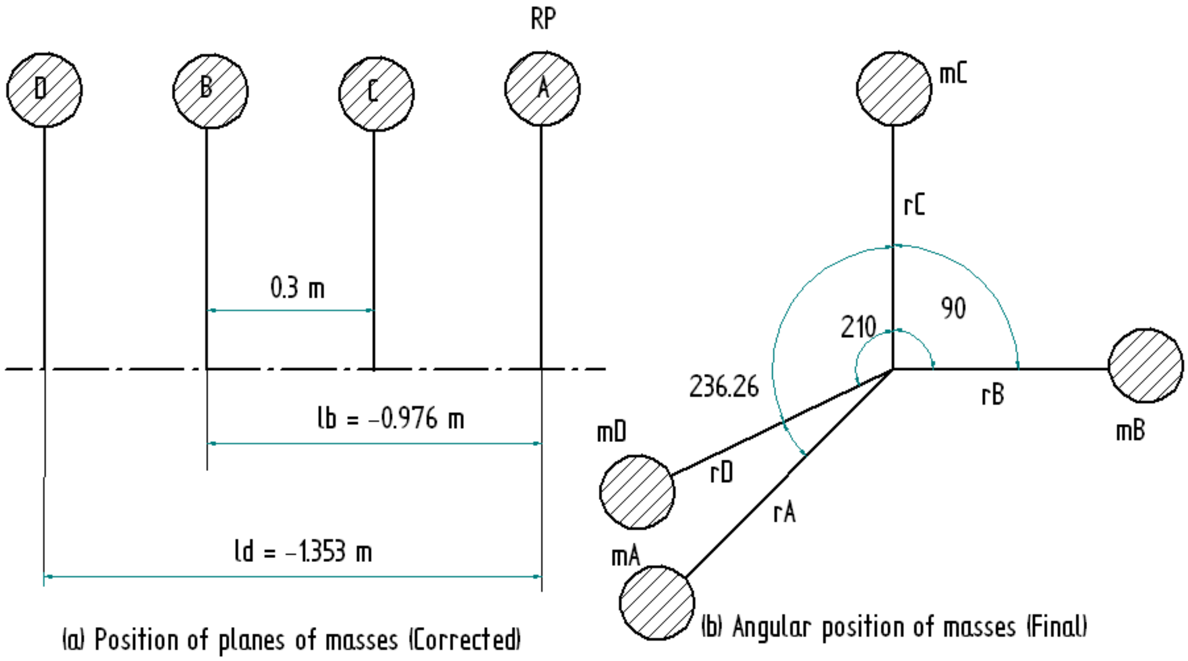
$$\mathbf{7.2l_B - 5.1962l_D = 0 \text{-----------} (3)}$$

$$\mathbf{6.0l_B} - 3\mathbf{l_D} \quad = \mathbf{1.8 \text{----------} (5)}$$

On solving the equations, we get

$\mathbf{l_D} = -1.353\,\text{m}$ and $\mathbf{l_B} = -0.976\,\text{m}$

As per the position of planes of masses assumed the distances shown are positive (+ ve) from the reference plane A. But the calculated values of distances l_B and l_D are negative. The corrected positions of planes of masses is shown below.



References:

1. Theory of Machines by S.S.Rattan, Third Edition, Tata McGraw Hill Education Private Limited.
2. Kinematics and Dynamics of Machinery by R. L. Norton, First Edition in SI units, Tata McGraw Hill Education Private Limited.
3. Primer on Dynamic Balancing "Causes, Corrections and Consequences" By
Jim Lyons International Sales Manager IRD Balancing Div. EntekIRD International

10 GEARS

Introduction

Gears are used to transmit motion from one shaft to another or between a shaft and a slide. This is accomplished by successively engaging teeth.

Gears use no intermediate link or connector and transmit the motion by direct contact. In this method, the surfaces of two bodies make a tangential contact. The two bodies have either a rolling or a sliding motion along the tangent at the point of contact. No motion is possible along the common normal as that will either break the contact or one body will tend to penetrate into the other.

If power transmitted between two shafts is small, motion between them may be obtained by using two plain cylinders or discs 1 and 2 as shown in Fig. 10.1. If there is no slip of one surface relative to the other, a definite motion of 1 can be transmitted to 2 and vice-versa. Such wheels are termed as *friction wheels*. However, as the power transmitted increases, slip occurs between the discs and the motion no longer remains definite.

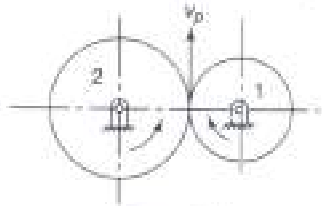


Fig. 10.1

Assuming no slipping of the two surfaces, the following kinematic relationship exists for their linear velocity:

$$v_p = \omega_1 r_1 = \omega_2 r_2$$
$$= 2\pi N_1 r_1 = 2\pi N_2 r_2$$

or
$$\frac{\omega_1}{\omega_2} = \frac{N_1}{N_2} = \frac{r_2}{r_1} \qquad (10.1)$$

where N = angular velocity (rpm)
ω = angular velocity (rad/s)
r = radius of the disc

Subscripts 1 and 2 represent discs 1 and 2 respectively.

The relationship shows that the speeds of the two discs rolling together without slipping are inversely proportional to the radii of the discs.

To transmit a definite motion of one disc to the other or to prevent slip between the surfaces, projections and recesses on the two discs can be made which can mesh with each other. This leads to the formation of teeth on the discs and the motion between the surfaces changes from rolling to sliding. The discs with teeth are known as *gears* or *gear wheels*.

It is to be noted that if the disc 1 rotates in the clockwise direction, 2 rotates in the counter-clockwise direction and vice-versa.

Although large velocity ratios of the driving and the driven members have been obtained by the use of gears, practically, it is limited to 6 for spur gears and 10 for helical and herringbone gears. To obtain large reductions, two or more pairs of gears are used.

10.1 CLASSIFICATION OF GEARS

Gears can be classified according to the relative positions of their shaft axes as follows:

1. Parallel Shafts

Regardless of the manner of contact, uniform rotary motion between two parallel shafts is equivalent to the rolling of two cylinders, assuming no slipping. Depending upon the teeth of the equivalent cylinders, i.e., straight or helical, the following are the main types of gears to join parallel shafts:

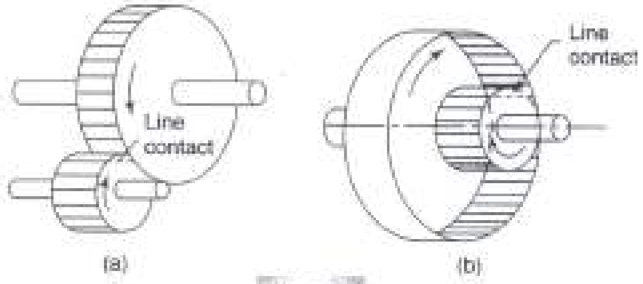


Fig. 10.2

Spur Gears They have straight teeth parallel to the axes and thus are not subjected to axial thrust due to tooth load [Fig. 10.2(a)].

At the time of engagement of the two gears, the contact extends across the entire width on a line parallel to the axes of rotation. This results in sudden application of the load, high impact stresses and excessive noise at high speeds.

Further, if the gears have external teeth on the outer surface of the cylinders, the shafts rotate in the opposite direction [Fig. 10.2(a)]. In an internal spur gear, the teeth are formed on the inner surface of an annulus ring. An internal gear can mesh with an external pinion (smaller gear) only and the two shafts rotate in the same direction as shown in [Fig. 10.2(b)].

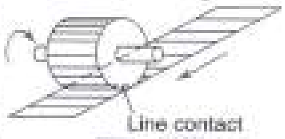


Fig. 10.3

Spur Rack and Pinion Spur rack is a special case of a spur gear where it is made of infinite diameter so that the pitch surface is a plane (Fig. 10.3). The spur rack and pinion combination converts rotary motion into translatory motion, or vice-versa. It is used in a lathe in which the rack transmits motion to the saddle.

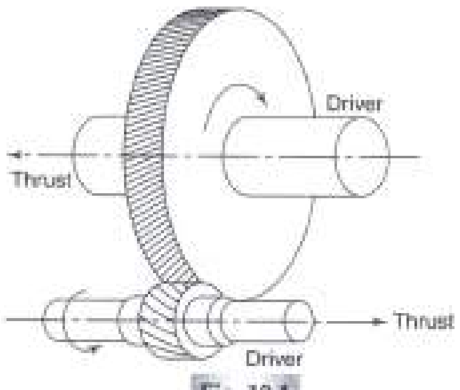


Fig. 10.4

Helical Gears or Helical Spur Gears In helical gears, the teeth are curved, each being helical in shape. Two mating gears have the same helix angle, but have teeth of opposite hands (Fig. 10.4).

At the beginning of engagement, contact occurs only at the point of leading edge of the curved teeth. As the gears rotate, the contact extends along a diagonal line across the teeth. Thus, the load application is gradual which results in low impact stresses and reduction in noise. Therefore, the helical gears can be used at higher velocities than the spur gears and have greater load-carrying capacity.

Helical gears have the disadvantage of having end thrust as there is a force component along the gear axis. The bearings and the assemblies mounting the helical gears must be able to withstand thrust loads.

Double-helical and Herringbone Gears A *double-helical* gear is equivalent to a pair of helical gears secured together, one having a right-hand helix and the other a left-hand helix. The teeth of the two rows are separated by a groove used for tool run out. Axial thrust which occurs in case of single-helical gears is eliminated in double-helical gears. This is because the axial thrusts of the two rows of teeth cancel each other out. These can be run at high speeds with less noise and vibrations.

If the left and the right inclinations of a double-helical gear meet at a common apex and there is no groove in between, the gear is known as *herringbone* gear (Fig. 10.5).

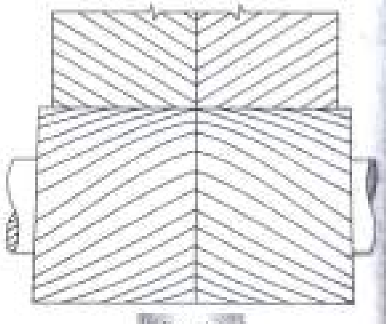


Fig. 10.5

2. Intersecting Shafts

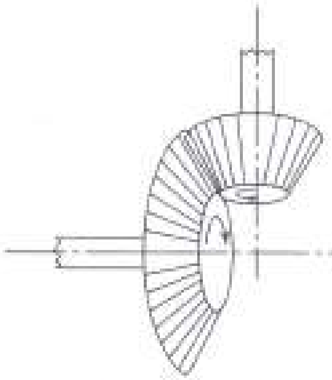


Fig. 10.6

Kinematically, the motion between two intersecting shafts is equivalent to the rolling of two cones, assuming no slipping. The gears, in general, are known as *bevel* gears.

When teeth formed on the cones are straight, the gears are known as *straight bevel* and when inclined, they are known as *spiral* or *helical bevel*.

Straight Bevel Gears The teeth are straight, radial to the point of intersection of the shaft axes and vary in cross section throughout their length. Usually, they are used to connect shafts at right angles which run at low speeds (Fig. 10.6). Gears of the same size and connecting two shafts at right angles to each other are known as *mitre* gears.

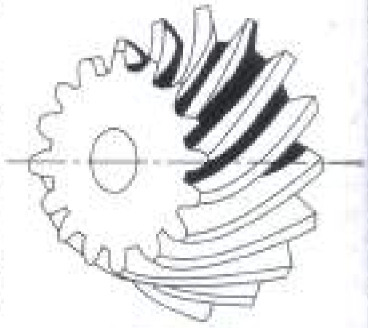


Fig. 10.7

At the beginning of engagement, straight bevel gears make the line contact similar to spur gears. There can also be *internal bevel* gears analogous to internal spur gears.

Spiral Bevel Gears When the teeth of a bevel gear are inclined at an angle to the face of the bevel, they are known as *spiral bevels* or *helical bevels* (Fig. 10.7). They are smoother in action and quieter than straight tooth bevels as there is gradual load application and low impact stresses. Of course, there exists an axial thrust calling for stronger bearings and supporting assemblies.

These are used for the drive to the differential of an automobile.

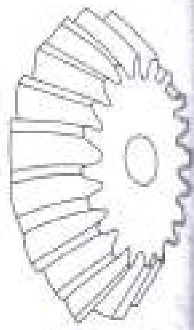


Fig. 10.8

Zerol Bevel Gears Spiral bevel gears with curved teeth but with a zero degree spiral angle are known as zerol bevel gears (Fig. 10.8). Their tooth action and the end thrust are the same as that of straight bevel gears and, therefore, can be used in the same mountings. However, they are quieter in action than the straight bevel type as the teeth are curved.

3. Skew Shafts

In case of parallel and intersecting shafts, a uniform rotary motion is possible by pure rolling contact. But in case of skew (non-parallel, non-intersecting) shafts, this is not possible.

Observe a hyperboloid shown in Fig. 10.9(a). It is a surface of revolution generated by a skew line *AB* revolving around an axis *O-O* in another plane, keeping the angle ψ_1 between them as constant. The minimum distance between AB and O-O is the common perpendicular CD which is also the radius of the gorge or throat of the hyperboloid.

As the generating element of a hyperboloid is a straight line, two hyperboloids can contact each other on a line common to their respective generating element, e.g., AB can be the generating element of the two hyperboloids [Fig. 10.9(b)]. Further, if the two mating hyperboloids are of limited width and have the rolling motion only, then contact length of their generators will go on diminishing and soon the two could be separated. In other words, if it is desired that the two hyperboloids touch each other on the entire length of AB as they roll, they must have some sliding motion parallel to the line of contact. Thus, if the two hyperboloids rotate on their respective axes, the motion between them would be a combination of rolling (normal to the line of contact) and sliding action (parallel to the line of contact). Teeth are cut on the hyperboloid surfaces parallel to the line of contact to form gears.

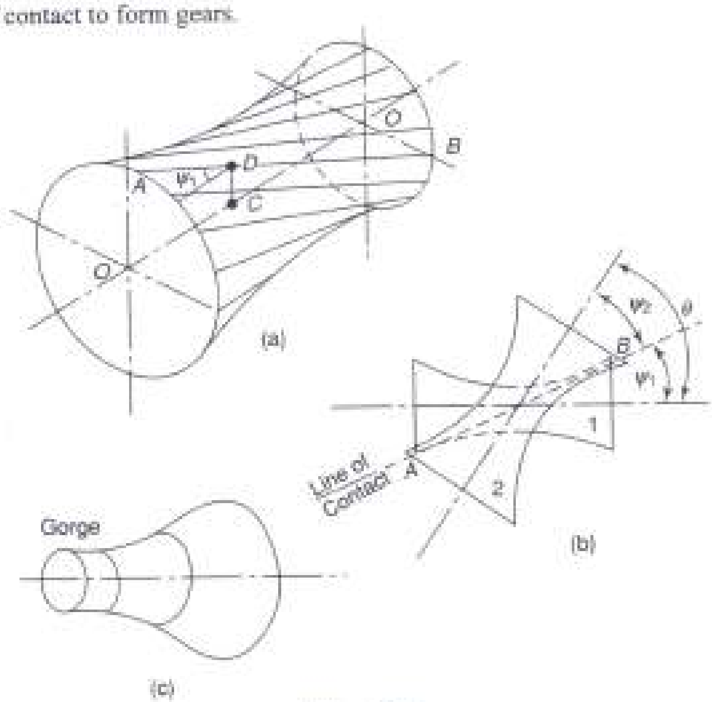


Fig. 10.9

Angle between the two shafts will be equal to the sum of the angles of generation of the two hyperboloids.

$$\theta = \psi_1 + \psi_2 \tag{10.2}$$

The minimum perpendicular distance between the two shafts is the sum of the gorge (throat) radii.

In practice, due to manufacturing difficulties, only portions of the hyperboloids are used to transmit motion between the skew shafts and that too with approximations as given below:

1. A short segment at the gorge is approximated to a cylinder and the corresponding gear is known as *helical* or *crossed-helical* or *spiral* gear [Fig. 10.9(c)]. The contact between the two gears is concentrated at a point which limits the capacity.

   For skew shafts with a 90° angle between them where high-speed ratios are to be achieved, the helix angle of the pinion (small gear) increases. When the angle exceeds 60°–65° and the number of teeth is less than 3–4, the high-speed pinion is known as *worm* and the mating helical gear as the *gear*.

2. Gears using an end portion of the hyperboloid are known as *hypoid* gears.

Thus, the main types of gears used for skew shafts are the following:

Crossed helical Gears The use of crossed-helical gears or spiral gears is limited to light loads. By a suitable choice of helix angle for the mating gears, the two shafts can be set at any angle (Fig. 10.10).

These gears are used to drive feed mechanisms on machine tools, camshafts and oil pumps on small IC engines, etc.

Worm Gears Worm gear is a special case of a spiral gear in which the larger wheel, usually, has a hollow or concave shape such that a portion of the pitch diameter of the other gear is enveloped on it. The smaller of the two wheels is called the worm which also has a large spiral angle.

The shafts may have any angle between them, but normally it is 90°. At least, one tooth of the worm must make a complete turn around the pitch cylinder and thus forms the screw thread. The sliding velocity of a worm gear is higher as compared to other types of gears.

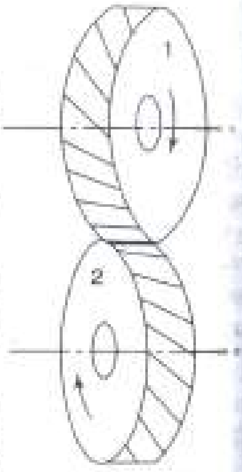


Fig. 10.10

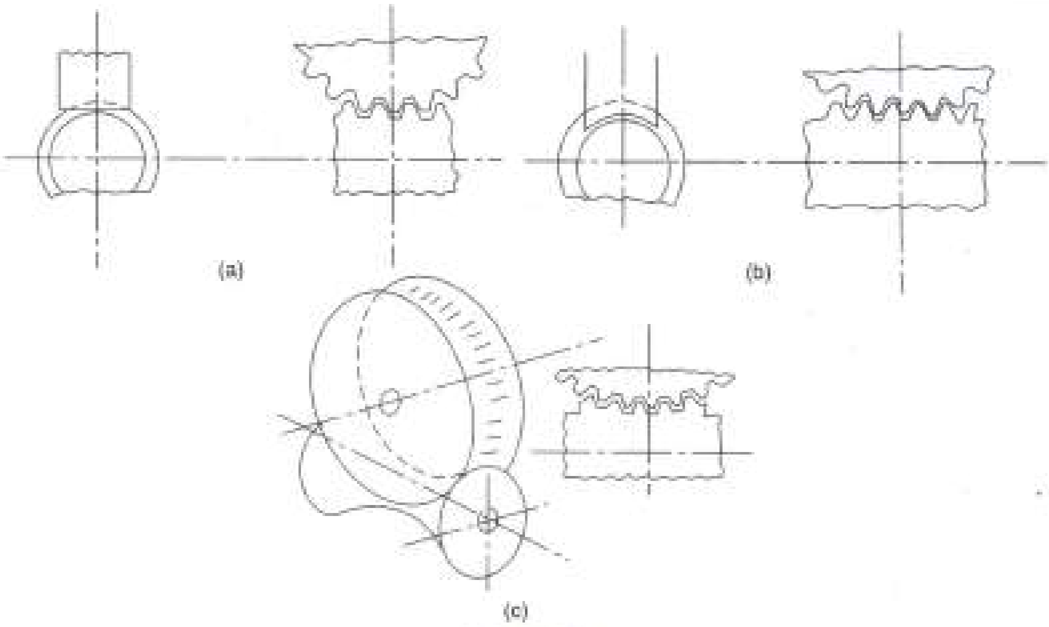


Fig. 10.11

Worm gears are made in the following forms:

1. *Non-throated (Fig. 10.11a)* The contact between the teeth is concentrated at a point.
2. *Single-throated (Fig. 10.11b)* Gear teeth are curved to envelop the worm. There is line contact between the teeth.
3. *Double-throated (Fig. 10.11c)* There is area contact between the teeth. A worm may be cut with a single- or a multiple-thread cutter.

Hypoid Gears As mentioned earlier, hypoid gears are approximations of hyperboloids though they look like spiral gears [Fig. 10.12(a)]. A hypoid pinion is larger and stronger than a spiral bevel pinion. A hypoid pair has a quiet and smooth action. Moreover, the shafts can pass each other so that bearings can be used on both sides of the gear and the pinion [Fig. 10.12(b)].

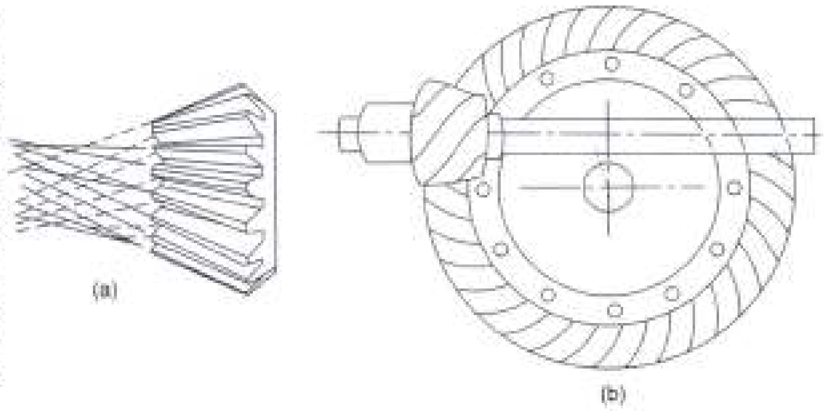


Fig. 10.12

There is continuous pitch line contact of the two mating hypoid gears while in action and they have larger number of teeth in contact than straight-tooth bevel gears. These can wear well if properly lubricated.

10.2 GEAR TERMINOLOGY

Various terms used in the study of gears have been explained below:

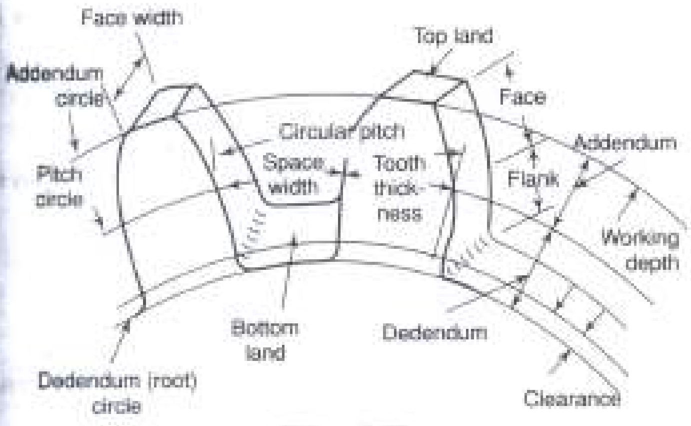


Fig. 10.13

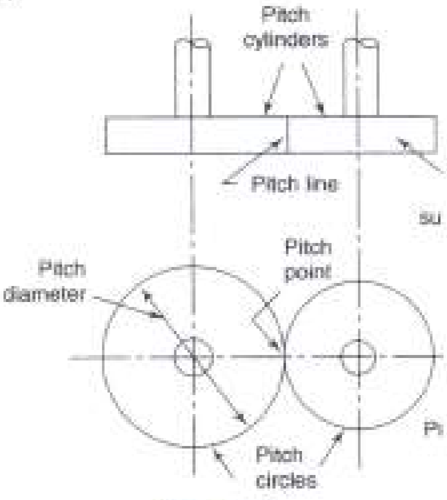


Fig. 10.14

1. Refer Figs 10.13 and 10.14.
   (a) *Pitch Cylinders* Pitch cylinders of a pair of gears in mesh are the imaginary friction cylinders, which by pure rolling together, transmit the same motion as the pair of gears.
   (b) *Pitch Circle* It is the circle corresponding to a section of the equivalent pitch cylinder by a plane normal to the wheel axis.

(c) *Pitch Diameter* It is the diameter of the pitch cylinder.

(d) *Pitch Surface* It is the surface of the pitch cylinder.

(e) *Pitch Point* The point of contact of two pitch circles is known as the pitch point.

(f) *Line of Centres* A line through the centres of rotation of a pair of mating gears is the line of centres.

(g) *Pinion* It is the smaller and usually the driving gear of a pair of mated gears.

2. (a) *Rack* It is a part of a gear wheel of infinite diameter (Fig. 10.15).

(b) *Pitch Line* It is a part of the pitch circle of a rack and is a straight line (Fig. 10.15).

Fig. 10.15

3. *Pitch* It is defined as follows:

(a) *Circular Pitch (p)* It is the distance measured along the circumference of the pitch circle from a point on one tooth to the corresponding point on the adjacent tooth (Fig. 10.13).

$$p = \frac{\pi d}{T}$$

where p = circular pitch
d = pitch diameter
T = number of teeth

As the expression for p involves π, an indeterminate number, p, cannot be expressed precisely. The angle subtended by the circular pitch at the centre of the pitch circle is known as the *pitch angle* (γ).

(b) *Diametral Pitch (P)* It is the number of teeth per unit length of the pitch circle diameter in inches.

$$P = \frac{T}{d}$$

The limitations of the diametral pitch is that it is not in terms of units of length, but in terms of teeth per unit length.
Also, it can be seen that

$$pP = \frac{\pi d}{T} \cdot \frac{T}{d} = \pi$$

The term *diametral pitch* is not used in SI units.

(c) *Module (m)* It is the ratio of the pitch diameter in mm to the number of teeth. The term is used in SI units in place of diametral pitch.

$$m = \frac{d}{T}$$

Also,

$$p = \frac{\pi d}{T} = \pi m$$

Pitch of two mating gears must be same.

4. (a) *Gear Ratio (G)* It is the ratio of the number of teeth on the gear to that on the pinion.

$$G = \frac{T}{t}$$

where T = number of teeth on the gear

t = number of teeth on the pinion.

(b) *Velocity Ratio (VR)* The velocity ratio is defined as the ratio of the angular velocity of the follower to the angular velocity of the driving gear.

Let d = pitch diameter

T = number of teeth

ω = angular velocity (rad/s)

N = angular velocity (rpm)

Subscript 1 = driver

2 = follower

$$\text{VR} = \frac{\text{angular velocity of follower}}{\text{angular velocity of driver}}$$

$$= \frac{\omega_2}{\omega_1}$$

$$= \frac{N_2}{N_1} \qquad (\omega = 2\pi N)$$

$$= \frac{d_1}{d_2} \qquad (\because \pi d_1 N_1 = \pi d_2 N_2)$$

$$= \frac{T_1}{T_2} \qquad \left(p = \frac{\pi d_1}{T_1} = \frac{\pi d_2}{T_2}\right) \qquad (10.3)$$

5. Refer to Fig. 10.13.

(i) (a) *Addendum Circle* It is a circle passing through the tips of teeth.

(b) *Addendum* It is the radial height of a tooth above the pitch circle. Its standard value is one module.

(c) *Dedendum* or *Root Circle* It is a circle passing through the roots of the teeth.

(d) *Dedendum* It is the radial depth of a tooth below the pitch circle. Its standard value is 1.157 m.

(e) *Clearance* Radial difference between the addendum and the dedendum of a tooth. Thus,

Addendum circle diameter = $d + 2m$

Dedendum circle diameter = $d - 2 \times 1.157\ m$

Clearance = $1.157\ m - m$

$= 0.157m$

(ii) (a) *Full Depth of Teeth* It is the total radial depth of the tooth space.

Full depth = Addendum + Dedendum

(b) *Working Depth of Teeth* The maximum depth to which a tooth penetrates into the tooth space of the mating gear is the working depth of teeth.

Working depth = Sum of addendums of the two gears.

(c) *Space Width* It is the width of the tooth space along the pitch circle.

(d) *Tooth Thickness* It is the thickness of the tooth measured along the pitch circle.

(e) *Backlash* It is the difference between the space width and the tooth thickness along the pitch circle. Backlash = Space width −Tooth thickness

(f) *Face Width* The length of the tooth parallel to the gear axis is the face width.

(iii) (a) *Top Land* It is the surface of the top of the tooth.

(b) *Bottom Land* The surface of the bottom of the tooth between the adjacent fillets.

(c) *Face* Tooth surface between the pitch circle and the top land.

(d) *Flank* Tooth surface between the pitch circle and the bottom land including fillet.

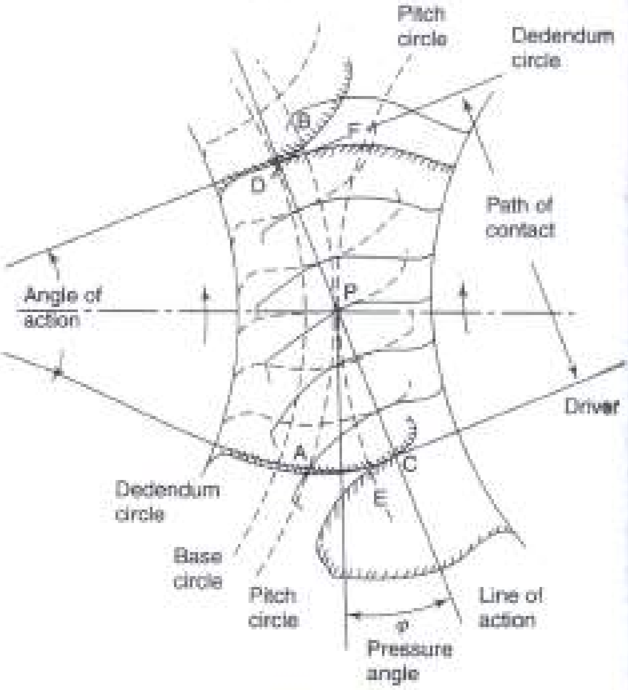


Fig. 10.16

(e) *Fillet* It is the curved portion of the tooth flank at the root circle.

6. Refer Fig. 10.16

(i) (a) *Line of Action or Pressure Line* The force, which the driving tooth exerts on the driven tooth, is along a line from the pitch point to the point of contact of the two teeth. This line is also the common normal at the point of contact of the mating gears and is known as the line of action or the pressure line.

(b) *Pressure Angle or Angle of Obliquity (φ)* The angle between the pressure line and the common tangent to the pitch circles is known as the pressure angle or the angle of obliquity.

For more power transmission and lesser pressure on the bearings, the pressure angle must be kept small. Standard pressure angles are 20° and 25°. Gears with 14.5° pressure angles have become almost obsolete.

(ii) (a) *Path of Contact or Contact Length* The locus of the point of contact of two mating teeth from the beginning of engagement to the end of engagement is known as the path of contact or the contact length. It is *CD* in the figure. The pitch point *P* is always one point on the path of contact. It can be subdivided as follows:

Path of Approach Portion of the path of contact from the beginning of engagement to the pitch point, i.e., the length *CP*.

Path of Recess Portion of the path of contact from the pitch point to the end of engagement, i.e., length *PD*.

(b) *Arc of Contact* The locus of a point on the pitch circle from the beginning to the end of engagement of two mating gears is known as the arc of contact. In Fig. 10.16, *APB* or *EPF* is the arc of contact.

It has also been divided into sub-portions.

Arc of Approach It is the portion of the arc of contact from the beginning of engagement to the pitch point, i.e., length *AP* or *EP*.

Arc of Recess The portion of the arc of contact from the pitch point to the end of engagement is the arc of recess, i.e., length *PB* or *PF*.

(c) *Angle of Action* (δ) It is the angle turned by a gear from the beginning of engagement to the end of engagement of a pair of teeth, i.e., the angle turned by arcs of contact of respective gear wheels.

Similarly, the angle of approach (α) and angle of recess (β) can be defined.

$$\delta = \alpha + \beta$$

The angle will have different values for the driving and the driven gears.

7. *Contact Ratio* It is the angle of action divided by the pitch angle, i.e.,

$$\text{Contact ratio} = \frac{\delta}{\gamma} = \frac{\alpha + \beta}{\gamma}$$

As the angle of action is the angle subtended by arc of contact and the pitch angle is the angle subtended by the circular pitch at the centre of the pitch circle, contact ratio is also the ratio of the arc of contact to the circular pitch, i.e.,

$$\text{Contact ratio} = \frac{\text{Arc of contact}}{\text{Circular pitch}}$$

Example 10.1 *Two spur gears have a velocity ratio of 1/3. The driven gear has 72 teeth of 8 mm module and rotates at 300 rpm. Calculate the number of teeth and the speed of the driver. What will be the pitch line velocities?*

Solution $T_2 = 72$; $VR = 1/3$; $N_2 = 300$ rpm; $m = 8$ mm

(i) $VR = \frac{N_2}{N_1} = \frac{T_1}{T_2} = \frac{1}{3}$ or $\frac{300}{N_1} = \frac{1}{3}$

or $N_1 = 900$ rpm

Also $\frac{T_1}{72} = \frac{1}{3}$ or $T_1 = 24$

(ii) Pitch line velocity, $v_p = \omega_1 r_1$ or $\omega_2 r_2$

$$= 2\pi N_1 \times \frac{d_1}{2} \text{ or } 2\pi N_2 \times \frac{d_2}{2}$$

$$= 2\pi N_1 \times \frac{mT_1}{2} \text{ or } 2\pi N_2 \times \frac{mT_2}{2}$$

$$= 2\pi \times 900 \times \frac{8 \times 24}{2} \text{ or } 2\pi \times 300 \times \frac{8 \times 72}{2}$$

$= 542\,867$ mm/minute

$= 9047.8$ mm/s or 9.0478 m/s

Example 10.2 *The number of teeth of a spur gear is 30 and it rotates at 200 rpm. What will be its circular pitch and the pitch line velocity if it has a module of 2 mm?*

Solution $T = 30$; $m = 2$ mm; $N = 200$ rpm

$p = \pi m = \pi \times 2 = \underline{6.28 \text{ mm}}$

$$v_p = \omega r = 2\pi N \times \frac{d}{2} = 2\pi N \times \frac{mT}{2}$$

$= \pi \times 200 \times 2 \times 30$

$= 37699$ mm / min $\quad = 628.3$ mm/s

10.3 LAW OF GEARING

The law of gearing states the condition which must be fulfilled by the gear tooth profiles to maintain a constant angular velocity ratio between two gears. Figure 10.17 shows two bodies 1 and 2 representing a portion of the two gears in mesh.

A point C on the tooth profile of the gear 1 is in contact with a point D on the tooth profile of the gear 2. The two curves in contact at points C or D must have a common normal at the point. Let it be $n-n$.

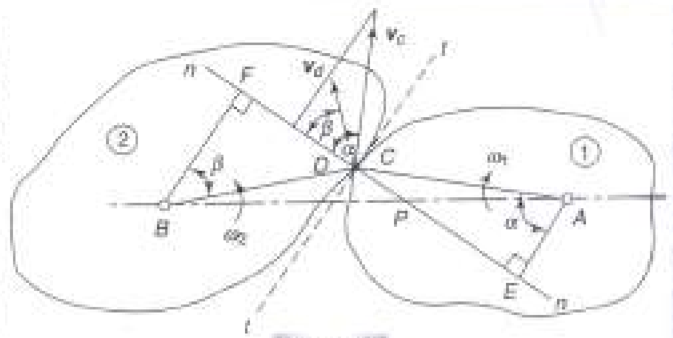


Fig. 10.17

Let ω_1 = instantaneous angular velocity of the gear 1 (clockwise)

ω_2 = instantaneous angular velocity of the gear 2 (counter-clockwise)

v_c = linear velocity of C

v_d = linear velocity of D

Then $v_c = \omega_1 . AC$ in a direction perpendicular to AC or at an angle α to $n-n$.

$v_c = \omega_2 BD$ in a direction perpendicular to BD or at an angle β to $n-n$.

Now, if the curved surfaces of the teeth of two gears are to remain in contact, one surface may slide relative to the other along the common tangent $t-t$. The relative motion between the surfaces along the common normal $n-n$ must be zero to avoid the separation, or the penetration of the two teeth into each other.

Component of v_c along $n-n = v_c \cos\alpha$

Component of v_d along $n-n = v_d \cos\beta$

Relative motion along $n-n = v_c \cos\alpha - v_d \cos\beta$

Draw perpendiculars AE and BF on $n-n$ from points A and B respectively. Then $\angle CAE = \alpha$ and $\angle DBF = \beta$. For proper contact,

$$v_c \cos\alpha - v_d \cos\beta = 0$$

or $$\omega_1 AC \cos\alpha - \omega_2 BD \cos\beta = 0$$

or $$\omega_1 AC \frac{AE}{AC} - \omega_2 BD \frac{BF}{BD} = 0$$

or $$\omega_1 AE - \omega_2 BF = 0$$

or $$\frac{\omega_1}{\omega_2} = \frac{BF}{AE}$$

$$= \frac{BP}{AP} \qquad [\because \Delta AEP \text{ and } BEP \text{ are similar}]$$

Thus, it is seen that the centre line AB is divided at P by the common normal in the inverse ratio of the angular velocities of the two gears. If it is desired that the angular velocities of two gears remain constant, the common normal at the point of contact of the two teeth should always pass through a fixed point P which divides the line of centres in the inverse ratio of angular velocities of two gears.

As seen earlier, P is also the point of contact of two pitch circles which divides the line of centres in the inverse ratio of the angular velocities of the two circles and is the pitch point.

Thus, for constant angular velocity ratio of the two gears, the common normal at the point of contact of the two mating teeth must pass through the pitch point.

Also, as the Δs AEP and BFP are similar,

$$\frac{BP}{AP} = \frac{FP}{EP}$$

or

$$\frac{\omega_1}{\omega_2} = \frac{FP}{EP} \quad \text{or} \quad \omega_1 EP = \omega_2 FP \tag{10.4}$$

10.4 VELOCITY OF SLIDING

If the curved surfaces of the two teeth of the gears 1 and 2 are to remain in contact, one can have a sliding motion relative to the other along the common tangent $t-t$ at C or D (Fig. 10.17).

Component of v_c along $t - t = v_c \sin\alpha$

Component of v_d along $t - t = v_d \sin\alpha$

Velocity of sliding $= v_c \sin\alpha - v_d \sin\beta$

$$= \omega_1 AC \frac{EC}{AC} - \omega_2 BD \frac{FD}{BD}$$

$$= \omega_1 EC - \omega_2 FD$$

$$= \omega_1 (EP + PC) - \omega_2 (FP - PD)$$

$$= \omega_1 EP + \omega_1 PC - \omega_2 FP + \omega_2 PC$$

(C and D are the coinciding points)

$$= (\omega_1 + \omega_2)\, PC + \omega_1 EP - \omega_2 FP$$

$$= (\omega_1 + \omega_2)\, PC \qquad [\omega_1 EP = \omega_2 FP, \text{ Eq. (10.4)}]$$

= sum of angular velocities × distance between the pitch point and the point of contact

10.5 FORMS OF TEETH

Two curves of any shape that fulfill the law of gearing can be used as the profiles of teeth. In other words, an arbitrary shape of one of the mating teeth can be taken and applying the law of gearing the shape of the other can be determined. Such gear are said to have *conjugate* teeth. However, it will be very difficult to manufacture such gears and the cost will be high. Moreover, on wearing, it will be very difficult to replace them with the available gears. Thus, there arises the need to standardize gear teeth.

Common forms of teeth that also satisfy the law of gearing are

1. Cycloidal profile teeth
2. Involute profile teeth

10.6 CYCLOIDAL PROFILE TEETH

In this type, the faces of the teeth are epicycloids and the flanks are the hypocycloids.

A *cycloid* is the locus of a point on the circumference of a circle that rolls without slipping on a fixed straight line.

An *epicycloid* is the locus of a point on the circumference of a circle that rolls without slipping on the circumference of another circle.

A *hypocycloid* is the locus of a point on the circumference of a circle that rolls without slipping inside the circumference of another circle.

The formation of a cycloidal tooth has been shown in Fig. 10.18. A circle H rolls inside another circle APB (pitch circle). At the start, the point of contact of the two circles is at A. As the circle H rolls inside the pitch circle, the locus of the point A on the circle H traces the path ALP which is a hypocycloid. A small portion of this curve near the pitch circle is used for the flank of the tooth.

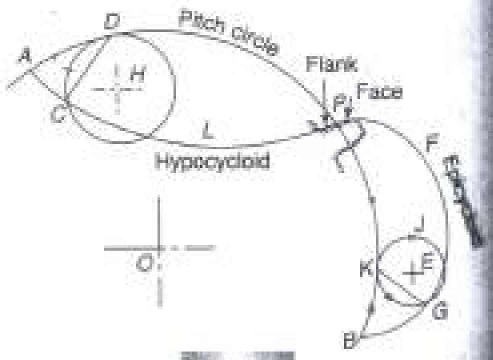


Fig. 10.18

A property of the hypocycloid is that at any instant, the line joining the generating point (A) with the point of contact of the two circles is normal to the hypocycloid, e.g., when the circle H touches the pitch circle at D, the point A is at C and CD is normal to the hypocycloid ALP.

Also, Arc AD = Arc CD (on circle H)

In the same way, if the circle E rolls outside the pitch circle, starting from P, an epicycloid PFB is obtained. Similar to the property of a hypocycloid, the line joining the generating point with the point of contact of the two circles is a normal to the epicycloid, e.g., when the circle E touches the pitch circle at K, the point P is at G and GK is normal to the epicycloid PFB.

Arc PK = Arc KJG (on circle E)

or Arc BK = Arc KG (on circle E)

A small portion of the curve near the pitch circle is used for the face of the tooth.

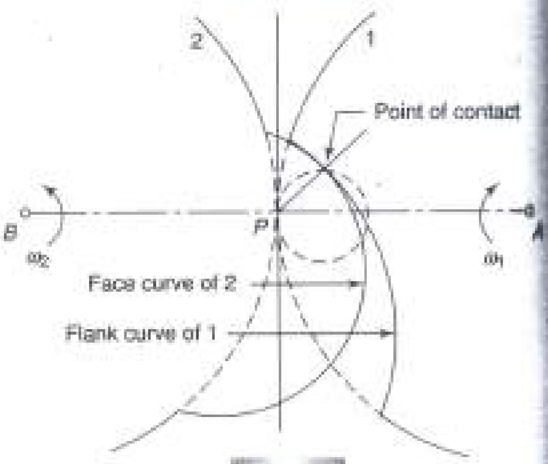


Fig. 10.19

Meshing of Teeth

During meshing of teeth, the face of a tooth on one gear is to mesh with the flank of another tooth on the other gear. Thus, for proper meshing, it is necessary that the diameter of the circle generating face of a tooth (on one gear) is the same as the diameter of the circle generating flank of the meshing tooth (on another gear); the pitch circle being the same in the two cases (Fig. 10.19).

Of course, the face and the flank of a tooth of a gear can be generated by two circles of different diameters. However, for interchangeability, the faces and flanks of both the teeth in the mesh are generated by the circles of the same diameter.

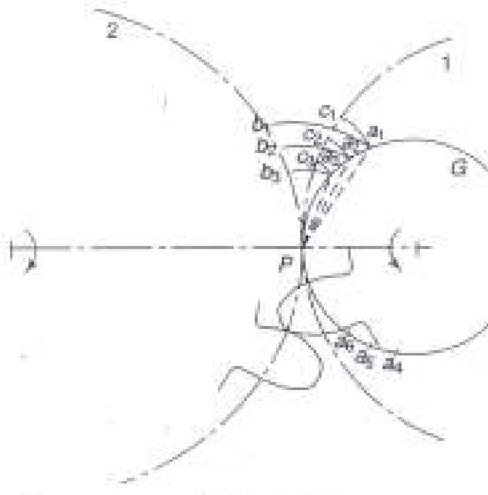


Fig. 10.20

Consider a generating circle G rolling outside the pitch circle of the gear 2 (Fig. 10.20). It will generate epicycloid, a portion of which is the face of tooth on the gear. Now this face is to mesh with the flank of a tooth on the gear 1. This flank will be a portion of the hypocycloid which can be generated by rolling the same generating circle G inside the pitch circle of the gear 1.

a_1 is the generating point for the two curves a_1b_1 (epicycloid) and a_1c_1 (hypocycloid). a_1b_1 is generated when the circle G moves in the clockwise direction on the pitch circle of the gear 2 and at the start a_1 coincides with b_1. a_1c_1 is generated when the circle G moves clockwise inside the pitch circle of the gear 2, and in the beginning a_1 coincides with c_1.

The two pitch circles touch each other at P (pitch point). When the generating circle G touches the pitch circle 2 at P, the generating point of the epicycloid is at a_1 and a_1P is normal to the face of tooth on the gear 2. Similarly, when G touches the pitch circle 1 at P, the generating point of the hypocycloid is again at a_1 and a_1P is also normal to the flank of tooth on the gear 1. Thus, if at an instant, a_1P is the common normal to the two profiles of the meshing teeth, the teeth must touch each other tangentially.

According to the law of gearing, the common normal at the point of contact of two mating profiles of the teeth must pass through a fixed point which is also the pitch point. The above discussion shows that the law of gearing is fulfilled in case of cycloidal teeth.

After a little while, let the point of contact of the two mating gears be at a_2. This point is on the generating circle G and if b_2 is considered the start of the epicycloid a_2b_2, and c_2 is considered the start of the hypocycloid a_2c_2 then a_2P will be normal to the two curves a_2b_2 and a_2c_2.

But as the two curves a_1b_1 and a_2b_2 are generated by the same circle rolling outside the same pitch circle, the two curves must be similar. Thus, a_2b_2 can be a portion of the curve a_1b_1. Similarly, a_2c_2 can be a portion of the curve a_1c_1.

Thus, in case of cycloidal teeth, the points of contact such as a_1, a_2, a_3, ..., P lie on the generating circle G.

After passing through the point P, the point of contact will shift on the other generating circle. Now, the flank of the tooth of the gear 1 will touch the face of the tooth of the gear 2. Thus, path of contact of cycloidal gears lies on the generating circles.

Path of approach = Arc $a_1a_2a_3P$

Arc of approach = Arc $b_1b_2b_3P$ = Arc $c_1c_2c_3P$

But arc $a_1a_2a_3P$ = Arc $b_1b_2b_3P$ = Arc $c_1c_2c_3P$

Therefore, the path of approach is equal to the arc of approach. In the same way, it can be shown that the path of contact will be equal to the arc of contact.

If the direction of rotation of the driver is reversed, the path of approach will be a_4, a_5, a_6 ....,P

Observe that in case of cycloidal teeth, the pressure angle varies from the maximum at the beginning of engagement to zero when the point of contact coincides with pitch point P and then again increases to maximum in the reverse direction.

As the common normal to the two meshing curves passes through the pitch point P, uniform rotary motion will be transmitted only as long as the pitch circles are tangent to each other. If the centre distance between the two pitch circles varies, the point P is shifted and the speed of the driven gear would vary.

Since the cycloidal teeth are made up of two curves, it is very difficult to produce accurate profiles. This has rendered this system obsolete.

10.7 INVOLUTE PROFILE TEETH

An *involute* is defined as the locus of a point on a straight line which rolls without slipping on the circumference of a circle. Also, it is the path traced out by the end of a piece of taut cord being unwound from the circumference of a circle. The circle on which the straight line rolls or from which the cord is unwound is known as the *base circle*.

Figure 10.21 shows an involute generated by a line rolling over the circumference of a base circle with centre at O. At the start, the tracing point is at A. As the line rolls on the circumference of the circle, the path ABC traced out by the point A is the involute.

Note that as D can be regarded as the instantaneous centre of rotation of B, the motion of B is perpendicular to BD. Since BD is tangent to the base circle, the normal to the involute is a tangent to the base circle.

A short length EF of the involute drawn from A can be utilized to make the profile of an involute tooth. The other side HJ of the tooth has been taken from the involute drawn from G in the reverse direction. The profile of an involute tooth is made up of a single curve, and teeth, usually, are termed as single curve teeth.

Owing to the ease of standardization and manufacture, and low cost of production, the use of involute teeth has become universal by entirely superseding the cycloidal shape. Only one cutter or tool is necessary to manufacture a complete set of interchangeable gears. The cutter is in the form of a rack as all gears will gear with their corresponding rack. Moreover, the cutters of this form can be made to a higher degree of accuracy as the teeth of an involute rack are straight.

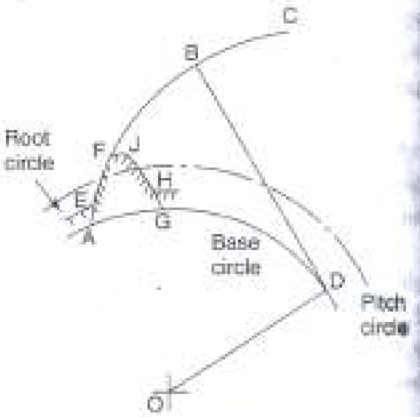
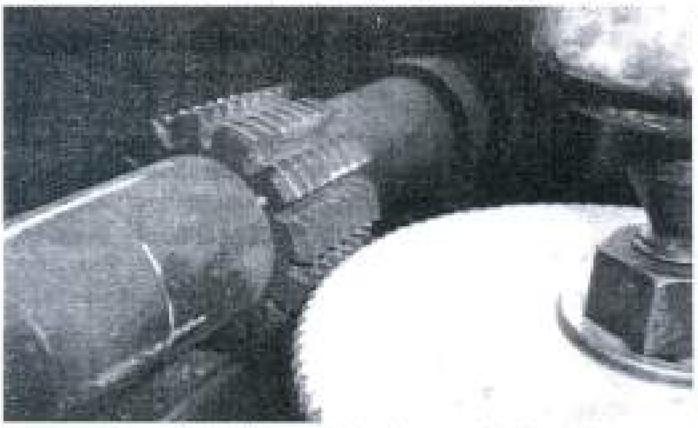


Fig. 10.21

Gear cutter of a milling machine. It cuts involute teeth.

Meshing of Teeth

In Fig. 10.22, two gear wheels 1 and 2 with centres of rotation at A and B respectively are in contact at C. CE and CF are the tangents to the two base circles 1 and 2 respectively. t-t is the common tangent to the two involutes DC and GCH of the two meshing teeth. The involute DC is traced by rolling line EF on the base circle of the gear 2 while the involute GCH is obtained by rolling line EF on the base circle of the gear 1.



Cutter of a hobbing machine. It cuts multiple teeth.

From the property of the involute, the tangent CF to the base circle of the gear 2 is normal to the involute DC or the tangent $t-t$. Similarly, the tangent CE to the base circle of the gear 1 is normal to the involute GC or the tangent $t-t$. As CE and CF both are normal to the common tangent t–t at the point C, CE and CF lie on a straight line. ECF is thus a straight line.

As the wheel 1 rotates in the clockwise direction, the point of contact C on the involute GCH pushes the involute DC along the line CF. Therefore, the path of contact of the two involute teeth is along the common tangent to the base circles. This common tangent is also the common normal to the two involutes at the point of contact for all positions.

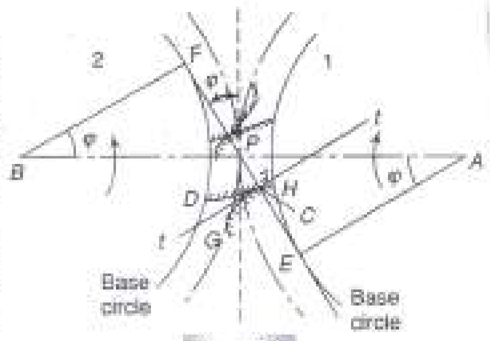


Fig. 10.22

Also, the common normal to the two involutes divides the line of centres of the two gears at P, the pitch point. Thus, the common normal always passes through the pitch point which is the point of contact of two pitch circles.

The line of action in case of involute teeth is along the common normal at the point of contact, which is fixed and is the common tangent to the two base circles. This shows that the pressure angle in this case remains constant throughout the engagement of the two teeth. The usual values of the pressure angles are 14.5°, 20° and 25°.

As EF is tangent to the base circle 1, AE is perpendicular to EF.

AEP is a right-angled triangle.

Also $\angle EAP = \varphi$

$AE = AP \cos \varphi$

Similarly, $BF = BP \cos\varphi$

i.e., [Base circle diameter = Pitch circle diameter × cosφ]

$$\text{velocity ratio of gears} = \frac{BP}{AP} = \frac{BF}{AE} = \text{constant}$$

Thus, for a pair of involute gears, the velocity ratio is inversely proportional to the pitch circle diameters as well as base circle diameters.

Any shift in the centres of two gears changes the centre distance. If the involutes are still in contact, the common normal to the two involutes at the point of contact will be the new common tangent to the base circles and its intersection with the line of centres as the new pitch point (Fig. 10.23). It can be judged that the shifting of P does not alter the ratio AP/BP which means the velocity ratio between the two gears remains constant. Of course, in this way there is change in the pressure angle. Altering the centre distance without destroying the correct tooth action is an important property of the involute gears.

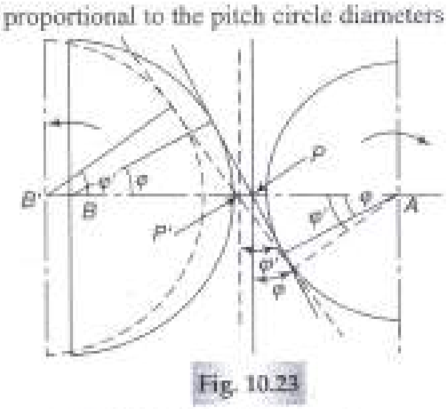


Fig. 10.23

Remember the following in case of involute gears:

1. Points of contact lie on the line of action which is the common tangent to the two base circles.
2. The contact is made when the tip of a tooth of the driven wheel touches the flank of a tooth of the driving wheel and the contact is broken when the tip of the driving wheel touches the flank of the driven wheel.
3. If the direction of angular movement of the wheels is reversed, the points of contact will lie on the other common tangent to the base circles.
4. Initial contact occurs where the addendum circle of the driven wheel intersects the line of action. Final contact occurs at a point where the addendum circle of the driver intersects the line of action.

10.8 INTERCHANGEABLE GEARS

The gears are interchangeable if they are standard ones. It is always a matter of convenience to have gears of standard dimensions which can be replaced easily when they are worn out. The gears are interchangeable if they have

- the same module,
- the same pressure angle,
- the same addendums and dedendums, and
- the same thickness.

A tooth system which relates the various parameters of gears such as pressure angle, addendum, dedendum, tooth thickness, working depth, etc., to attain interchangeability of the gears of all tooth numbers, but of the same pressure angle and pitch is said to be a *standard system*. Usually, the standard cutters are available for their manufacture.

In Table 10.1, tooth proportions for completely interchangeable gears are given. They can be used for operation on standard centre distances. The 14.5° pressure angle system has become obsolete now as the size of the gears used to be larger as compared to the gears with higher angles.

Table 10.1

Tooth system	Pressure angle	addendum	dedendum
Full depth	20°	1 m	1.25 m or 1.35 m
	22.5°	1 m	1.25 m or 1.35 m
	25°	1 m	1.25 m or 1.35 m
Stub	20°	0.8 m	1 m

Preferred modules: 1, 1.25, 1.5, 2, 2.5 3, 4, 5, 6, 8, 10, 12, 16, 20, 25, 30, 40, 50

10.9 NON-STANDARD GEARS

The term non-standard gears apply to such gears as are modified by changing some standard parameters like pressure angle, addendum, tooth depth or centre distance. These changes are made to improve the performance of the gear operation or from the economical point of view.

The recent trend these days is to make the designs of machines as compact as possible to reduce their size and weight which also results in reduction in the costs. Consider a gear set to have a velocity ratio of 4:1. If a pinion of 80 mm pitch diameter is selected for the purpose, the pitch diameter of the gear is 320 mm. Thus, space requirement of the gear is 400 mm. Now, if somehow the pitch diameter of the pinion is reduced by 10 mm, the pitch diameter of the gear is reduced by 40 mm, and the overall reduction in space is 50 mm. Also, the sizes of other components associated with the gear set such as shafts, casings and bearings are also reduced. The only way to have a smaller size of gears is to reduce the number of teeth. However, for a typical type of teeth, it is observed that if the number of teeth is reduced from a certain number, the problems of interference, undercutting and contact ratio hamper the smooth running of the gears. Therefore, the main reason to employ non-standard gears is to prevent interference and undercutting and to maintain a reasonable contact ratio.

It should be remembered that as an involute is generated, its radius of curvature goes on becoming larger and larger, being zero at the base circle. As far as possible, the curve near the base should be avoided because high stresses are developed in the region of sharp curvature.

Centre-distance Modifications The number of teeth on a pinion can be reduced from the minimum allowable number by increasing the centre distance marginally and by changing the tooth proportions and the pressure angle of the gears. A reduction in the interference and improvement in the contact ratio is brought this way. The teeth can be generated with rack cutters of standard pressure angles by displacing the pitch line of the rack from the pitch circle of the gear. This action produces teeth which are thicker than before. As the teeth are cut with a displaced or offset cutter, they will engage at a new pressure angle and at a new centre distance.

Clearance Modifications If the clearance between mating teeth is increased to 0.3 *m* or 0.4 *m* instead of the usual value of 0.25 *m* to have a larger fillet at the root of the tooth, the fatigue strength of the tooth is increased. This way some extra depth is available to smoothen the tooth profile. Interchangeability is not lost this way.

Addendum Modifications In cases where it is not possible to change the centre distances, modifications can be made to the addendum. In such cases, there has to be no change in the pitch circles and the pressure angles. However, the contact region is shifted away from the pinion centre towards the gear centre, decreasing the approach action and increasing the recess action.

Example 10.3 *The following data relate to two meshing gears Velocity ratio* $= \frac{1}{3}$, *Module = 4 mm, Pressure angle = 20°, Centre distance = 200 mm*

Determine the number of teeth and the base circle radius of the gear wheel.

Solution $VR = 1/3$, $\varphi = 20°$, $m = 4$ mm
$C = 200$ mm

(i) $VR = \frac{N_2}{N_1} = \frac{1}{3} = \frac{T_1}{T_2}$ or $T_2 = 3T_1$

and $C = \frac{d_1 + d_2}{2} = \frac{m\,(T_1 + T_2)}{2}$

or $200 = \frac{4\,(T_1 + 3T_1)}{2} = 8T_1$

or $T_1 = 25$ and $T_2 = 25 \times 3 = 75$

Number of teeth on gear wheel = 75

(ii) $d_2 = m\,T_2 = 4 \times 75 = 300$ mm

Base circle radius, $d_{b2} = \frac{d_2}{2} \cos\varphi$

$= \frac{300}{2} \times \cos 20° = 141$ mm

10.10 PATH OF CONTACT

Let two gear wheels with centres *A* and *B* be in contact (Fig. 10.24).

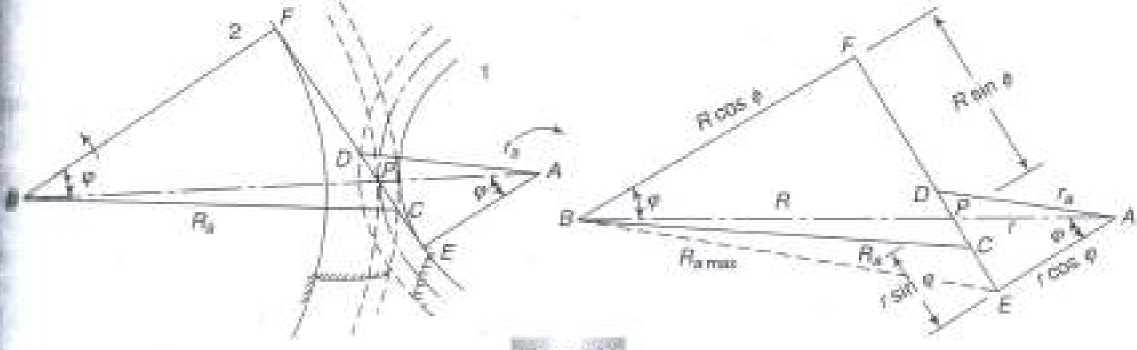


Fig. 10.24

The pinion 1 is the driver and is rotating clockwise. The wheel 2 is driven in the counter-clockwise direction. EF is their common tangent to the base circles.

Contact of the two teeth is made where the addendum circle of the wheel meets the line of action EF, i.e., at C and is broken where the addendum circle of the pinion meets the line of action, i.e., at D. CD is then the path of contact.

Let r = pitch circle radius of pinion

R = pitch circle radius of wheel

r_a = addendum circle radius of pinion

R_a = addendum circle radius of wheel.

Path of contact = path of approach + path of recess

$$CD = CP + PD$$
$$= (CF - PF) + (DE - PE)$$
$$= \left[\sqrt{R_a^2 - R^2\cos^2\varphi} - R\sin\varphi\right] + \left[\sqrt{r_a^2 - r^2\cos^2\varphi} - r\sin\varphi\right]$$
$$= \sqrt{R_a^2 - R^2\cos^2\varphi} + \sqrt{r_a^2 - r^2\cos^2\varphi} - (R + r)\sin\varphi \quad (10.5)$$

Observe that the path of approach can be found if the dimensions of the driven wheel are known. Similarly, the path of recess is known from the dimensions of the driving wheel (pinion).

10.11 ARC OF CONTACT

The arc of contact is the distance travelled by a point on either pitch circle of the two wheels during the period of contact of a pair of teeth.

In Fig.10.25, at the beginning of engagement, the driving involute is shown as GH; when the point of contact is at P, it is shown as JK and when at the end of engagement, it is DL. The arc of contact is $P'P''$ and it consists of the arc of approach $P'P$ and the arc of recess PP''.

Let the time to traverse the arc of approach is t_a. Then

Arc of approach = $P'P$ = Tangential velocity of P' × Time of approach

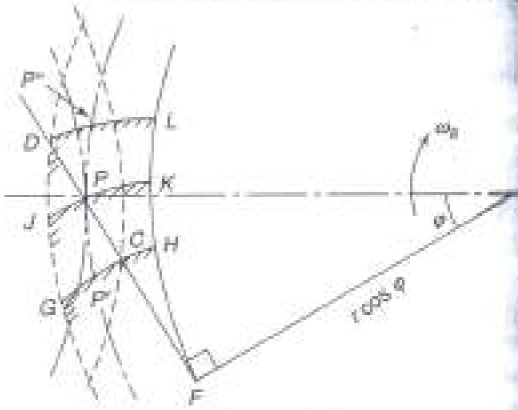


Fig. 10.25

$$= \omega_a r \times t_a \quad (t_a = \text{time of approach})$$
$$= \omega_a (r\cos\varphi)\frac{1}{\cos\varphi} t_a$$
$$= (\text{Tang. vel. of } H)t_a \frac{1}{\cos\varphi} \quad (AF = AH)$$
$$= \frac{\text{Arc } HK}{\cos\varphi}$$

$$= \frac{\text{Arc } FK - \text{Arc } FH}{\cos\varphi}$$

$$= \frac{FP - FC}{\cos\varphi} = \frac{CP}{\cos\varphi}$$

Arc FK is equal to the path FP as the point P is on the generator FP that rolls on the base circle FHK to generate the involute PK. Similarly, arc FH = Path FC.

Arc of recess = PP'' = Tang. vel. of P × Time of recess

$$= \omega_a r \times t_r \qquad (t_r = \text{time of recess})$$

$$= \omega_a (r\cos\varphi)\frac{1}{\cos\varphi} t_r$$

$$= (\text{Tang. vel. of } K)\, t_r \frac{1}{\cos\varphi}$$

$$= \frac{\text{Arc } KL}{\cos\varphi} = \frac{\text{Arc } FL - \text{Arc } FK}{\cos\varphi}$$

or

$$PP'' = \frac{FD - FP}{\cos\varphi} = \frac{PD}{\cos\varphi}$$

$$\text{Arc of contact} = \frac{CP}{\cos\varphi} + \frac{PD}{\cos\varphi} = \frac{CP + PD}{\cos\varphi} = \frac{CD}{\cos\varphi}$$

or

$$\text{Arc of contact} = \frac{\text{Path of contact}}{\cos\varphi} \tag{10.6}$$

10.12 NUMBER OF PAIRS OF TEETH IN CONTACT (CONTACT RATIO)

The arc of contact is the length of the pitch circle traversed by a point on it during the mating of a pair of teeth.

Thus, all the teeth lying in between the arc of contact will be meshing with the teeth on the other wheel.

Therefore, the number of teeth in contact, $n = \dfrac{\text{Arc of contact}}{\text{Circular pitch}} = \dfrac{CD}{\cos\varphi}\dfrac{1}{p}$ (10.7)

As the ratio of the arc of contact to the circular pitch is also the *contact ratio*, the number of teeth is also expressed in terms of contact ratio.

For continuous transmission of motion, at least one tooth of one wheel must be in contact with another tooth of the second wheel. Therefore, n must be greater than unity.

If n lies between 1 and 2, the number of teeth in contact at any time will not be less than one and never more than two. If n is between 2 and 3, it is never less than two pairs of teeth and not more than three pairs, and so on. If n is 1.6, one pair of teeth are always in contact whereas two pairs of teeth are in contact for 60% of the time.

Example 10.4 *Each of two gears in a mesh has 48 teeth and a module of 8 mm. The teeth are of 20° involute profile. The arc of contact is 2.25 times the circular pitch. Determine the addendum.*

Solution $\varphi = 20°$; $t = T = 48$; $m = 8$ mm;

$$R = r = \frac{mT}{2} = \frac{8 \times 48}{2} = 192 \text{ mm}; \; R_a = r_a$$

Arc of contact = 2.25 × Circular pitch = $2.25\pi m$
$= 2.25\pi \times 8 = 56.55$ mm

Path of contact = $56.55 \times \cos 20° = 53.14$ mm

$$\text{or} \quad (\sqrt{R_a^2 - R^2\cos^2\varphi} - R\sin\varphi) + (\sqrt{r_a^2 - r^2\cos^2\varphi} - r\sin\varphi) = 53.14$$

$$\text{or} \quad 2(\sqrt{R_a^2 - 192^2\cos^2 20°} - 192\sin 20°) = 53.14 \quad \text{or} \quad R_a = 202.6 \text{ mm}$$

Addendum = $R_a - R = 202.6 - 192 = 10.6$ mm

Example 10.5 *Two involute gears in mesh have 20° pressure angle. The gear ratio is 3 and the number of teeth on the pinion is 24. The teeth have a module of 6 mm. The pitch line velocity is 1.5 m/s and the addendum equal to one module. Determine the angle of action of the pinion (the angle turned by the pinion when one pair of teeth is in the mesh) and the maximum velocity of sliding.*

Solution $\varphi = 20°$; $t = 24$; $m = 6$ mm;
$T = 24 \times 3 = 72$;

$$r = \frac{mt}{2} = \frac{6 \times 24}{2} = 72 \text{ mm};$$

$R = 72 \times 3 = 216$ mm; $r_a = 72 + 6 = 78$ mm;
$R_a = 216 + 6 = 222$ mm

$$\text{Path of contact} = (\sqrt{R_a^2 - R^2\cos^2\varphi} - R\sin\varphi) + (\sqrt{r_a^2 - r^2\cos^2\varphi} - r\sin\varphi)$$

$$= (\sqrt{222^2 - 216^2\cos^2 20°} - 216\sin 20°) + (\sqrt{78^2 - 72^2\cos^2 20°} - 72\sin 20°)$$

$= 16.04 + 14.18 = 30.22$ mm

$$\text{Arc of contact} = \frac{\text{Path of contact}}{\cos\varphi} = \frac{30.22}{\cos 20°}$$

$= 32.16$ mm

$$\text{Angle of action} = \frac{\text{Arc of contact}}{r} = \frac{32.16}{72}$$

$= 0.4467$ rad $= 0.4467 \times 180/\pi = 25.59°$

Velocity of sliding = $(\omega_p + \omega_g)$ × Path of approach

$$= \left(\frac{v}{r} + \frac{v}{R}\right) \times \text{Path of approach}$$

$$= \left(\frac{1500}{72} + \frac{1500}{216}\right) \times 16.04 = 445.6 \text{ mm/s}$$

Example 10.6 *Two involute gears in a mesh have a module of 8 mm and a pressure angle of 20°. The larger gear has 57 while the pinion has 23 teeth. If the addenda on pinion and gear wheels are equal to one module, find the*

(i) contact ratio (the number of pairs of teeth in contact)
(ii) angle of action of the pinion and the gear wheel
(iii) ratio of the sliding to rolling velocity at the
(a) beginning of contact
(b) pitch point
(c) end of contact

Solution $\varphi = 20°$; $T = 57$; $t = 23$; $m = 8$ mm;
addendum = m = 8 mm

$$R = \frac{mT}{2} = \frac{8 \times 57}{2} = 228 \text{ mm};$$

$$R_a = R + m = 228 + 8 = 236 \text{ mm}$$

$$r = \frac{mt}{2} = \frac{8 \times 23}{2} = 92 \text{ mm};$$

$r_a = r + m = 92 + 8 = 100$ mm

(i) $n = \dfrac{\text{Arc of contact}}{\text{Circular pitch}} = \left(\dfrac{\text{Path of contact}}{\cos\varphi}\right)$

$$\times \frac{1}{\pi m} = \frac{\text{Path of approach + Path of recess}}{\cos\varphi \times \pi m}$$

$$= \frac{\left[\begin{array}{l}\sqrt{R_a^2 - R^2\cos^2\varphi} - R\sin\varphi \\ +\sqrt{r_a^2 - r^2\cos^2\varphi} - r\sin\varphi\end{array}\right]}{\cos\varphi \times \pi m}$$

$$\frac{\left[\begin{array}{l}\sqrt{(236)^2 - (228^2\cos^2 20^\circ} - 228\sin 20^\circ \\ +\sqrt{(100)^2 - (92)^2\cos^2 20^\circ} - 92\sin 20^\circ\end{array}\right]}{\cos 20^\circ \pi \times 8}$$

$$= \frac{20.97 + 18.79}{\cos 20^\circ \times \pi \times 8} = 42.31 \times \frac{1}{\pi \times 8} = \underline{1.68}$$

(ii) Angle of action, $\delta_p = \dfrac{\text{Arc of contact}}{r} = \dfrac{42.31}{92}$

$= 0.46$ rad or $0.46 \times 180/\pi = 26.3^\circ$

$\delta_g = \dfrac{\text{Arc of contact}}{R} = \dfrac{42.31}{228} = 0.1856$ rad

or $0.1856 \times 180/\pi = 10.63^\circ$

(iii) (a) $\dfrac{\text{Sliding velocity}}{\text{Rolling velocity}}$

$$= \frac{(\omega_p + \omega_g) \times \text{Path of approach}}{\text{Pitch line velocity} (= \omega_p \times r)}$$

$$= \frac{\left(\omega_p + \frac{23}{57}\omega_p\right) \times 20.97}{\omega_p \times 92} = \underline{0.32}$$

(b) $\dfrac{\text{Sliding velocity}}{\text{Rolling velocity}} = \dfrac{(\omega_p + \omega_g) \times 0}{\text{Pitch line velocity}} = 0$

(c) $\dfrac{\text{Sliding velocity}}{\text{Rolling velocity}}$

$$= \frac{\left(\omega_p + \frac{23}{57}\omega_p\right) \times \text{Path of recess}}{\omega_p \times r}$$

$$= \frac{\left(1 + \frac{23}{57}\right) \times 18.79}{92} = \underline{0.287}$$

Example 10.7 *Two 20° gears have a module pitch of 4 mm. The number of teeth on gears 1 and 2 are 40 and 24 respectively. If the gear 2 rotates at 600 rpm, determine the velocity of sliding when the contact is at the tip of the tooth of gear 2. Take addendum equal to one module.*

Also, find the maximum velocity of sliding.

Solution 1 is the gear wheel and 2 is the pinion.

$\varphi = 20^\circ$; $T = 40$; $N_p = 600$ mm; $t = 24$; $m = 4$ mm

Addendum = 1 module = 4 mm

$R = \dfrac{mT}{2} = \dfrac{4 \times 40}{2} = 80$ mm; $R_a = 80 + 4 = 84$ mm

$r = \dfrac{mt}{2} = \dfrac{4 \times 24}{2} = 48$ mm; $r_a = 48 + 4 = 52$ mm

$N_g = N_p \times \dfrac{t}{T} = 600 \times \dfrac{24}{40} = 360$ rpm

(i) Let pinion (gear 2) be the driver.

The tip of the driving wheel is in contact with a tooth of the driven wheel at the end of engagement. Thus, it is required to find the path of recess which is obtained from the dimensions of the driving wheel.

Path of recess $= \sqrt{r_a^2 - (r\cos\varphi)^2} - r\sin\varphi$

$= \sqrt{(52)^2 - (48\cos 20^\circ)^2} - 48\sin 20^\circ$

$= 9.458$ mm

Velocity of sliding = $(\omega_p + \omega_g) \times$ Path of recess

$= 2\pi (N_p + N_g) \times 9.458$

$= 2\pi (600 + 360) \times 9.458$

$= 57\,049$ mm/min

$= \underline{950.8 \text{ mm/s}}$

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13

Gear Trains

Features

13.1. Introduction

Sometimes, two or more gears are made to mesh with each other to transmit power from one shaft to another. Such a combination is called ***gear train*** or ***train of toothed wheels.*** The nature of the train used depends upon the velocity ratio required and the relative position of the axes of shafts. A gear train may consist of spur, bevel or spiral gears.

13.2. Types of Gear Trains

Following are the different types of gear trains, depending upon the arrangement of wheels :

1. Simple gear train, **2.** Compound gear train, **3.** Reverted gear train, and **4.** Epicyclic gear train.

In the first three types of gear trains, the axes of the shafts over which the gears are mounted are fixed relative to each other. But in case of epicyclic gear trains, the axes of the shafts on which the gears are mounted may move relative to a fixed axis.

13.3. Simple Gear Train

When there is only one gear on each shaft, as shown in Fig. 13.1, it is known as ***simple gear train.*** The gears are represented by their pitch circles.

When the distance between the two shafts is small, the two gears 1 and 2 are made to mesh with each other to

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transmit motion from one shaft to the other, as shown in Fig. 13.1 (*a*). Since the gear 1 drives the gear 2, therefore gear 1 is called the ***driver*** and the gear 2 is called the ***driven*** or ***follower***. It may be noted that the motion of the driven gear is opposite to the motion of driving gear.

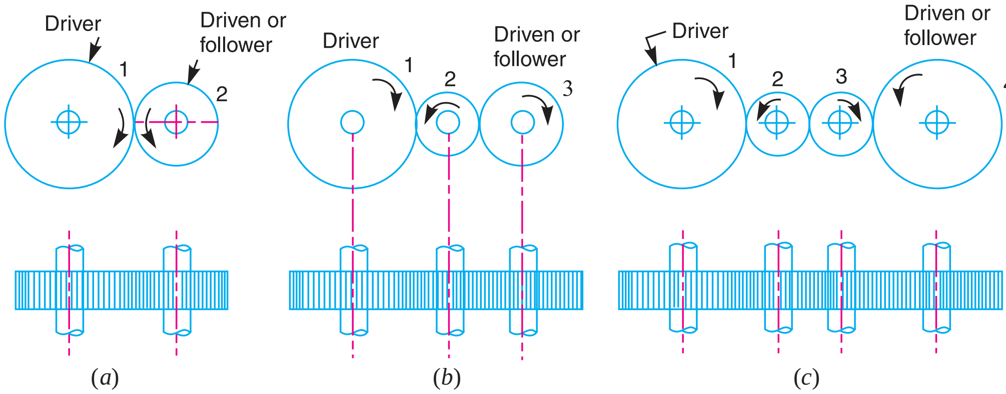


Fig. 13.1. Simple gear train.

Let N_1 = Speed of gear 1(or driver) in r.p.m.,

N_2 = Speed of gear 2 (or driven or follower) in r.p.m.,

T_1 = Number of teeth on gear 1, and

T_2 = Number of teeth on gear 2.

Since the speed ratio (or velocity ratio) of gear train is the ratio of the speed of the driver to the speed of the driven or follower and ratio of speeds of any pair of gears in mesh is the inverse of their number of teeth, therefore

$$\text{Speed ratio} = \frac{N_1}{N_2} = \frac{T_2}{T_1}$$

It may be noted that ratio of the speed of the driven or follower to the speed of the driver is known as ***train value*** of the gear train. Mathematically,

$$\text{Train value} = \frac{N_2}{N_1} = \frac{T_1}{T_2}$$

From above, we see that the train value is the reciprocal of speed ratio.

Sometimes, the distance between the two gears is large. The motion from one gear to another, in such a case, may be transmitted by either of the following two methods :

1. By providing the large sized gear, or **2.** By providing one or more intermediate gears.

A little consideration will show that the former method (*i.e*. providing large sized gears) is very inconvenient and uneconomical method ; whereas the latter method (*i.e*. providing one or more intermediate gear) is very convenient and economical.

It may be noted that when the number of intermediate gears are ***odd,*** the motion of both the gears (*i.e*. driver and driven or follower) is ***like*** as shown in Fig. 13.1 (*b*).

But if the number of intermediate gears are ***even,*** the motion of the driven or follower will be in the opposite direction of the driver as shown in Fig. 13.1 (*c*).

Now consider a simple train of gears with one intermediate gear as shown in Fig. 13.1 (*b*).

Let N_1 = Speed of driver in r.p.m.,

N_2 = Speed of intermediate gear in r.p.m.,

N_3 = Speed of driven or follower in r.p.m.,

T_1 = Number of teeth on driver,

T_2 = Number of teeth on intermediate gear, and

T_3 = Number of teeth on driven or follower.

Since the driving gear 1 is in mesh with the intermediate gear 2, therefore speed ratio for these two gears is

$$\frac{N_1}{N_2} = \frac{T_2}{T_1} \qquad \text{...}(i)$$

Similarly, as the intermediate gear 2 is in mesh with the driven gear 3, therefore speed ratio for these two gears is

$$\frac{N_2}{N_3} = \frac{T_3}{T_2} \qquad \text{...}(ii)$$

The speed ratio of the gear train as shown in Fig. 13.1 (*b*) is obtained by multiplying the equations (*i*) and (*ii*).

$$\therefore \qquad \frac{N_1}{N_2} \times \frac{N_2}{N_3} = \frac{T_2}{T_1} \times \frac{T_3}{T_2} \quad \text{or} \quad \frac{N_1}{N_3} = \frac{T_3}{T_1}$$

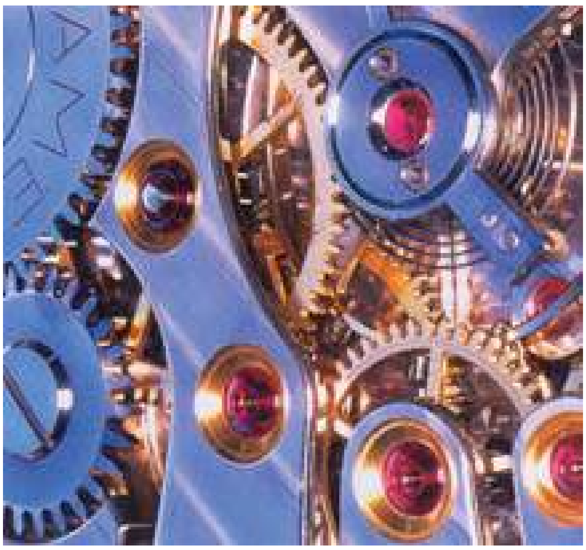
i.e.
$$\text{Speed ratio} = \frac{\text{Speed of driver}}{\text{Speed of driven}} = \frac{\text{No. of teeth on driven}}{\text{No. of teeth on driver}}$$

and
$$\text{Train value} = \frac{\text{Speed of driven}}{\text{Speed of driver}} = \frac{\text{No. of teeth on driver}}{\text{No. of teeth on driven}}$$

Similarly, it can be proved that the above equation holds good even if there are any number of intermediate gears. From above, we see that the speed ratio and the train value, in a simple train of gears, is independent of the size and number of intermediate gears. These intermediate gears are called ***idle gears,*** as they do not effect the speed ratio or train value of the system. The idle gears are used for the following two purposes :

1. To connect gears where a large centre distance is required, and

2. To obtain the desired direction of motion of the driven gear (*i.e.* clockwise or anticlockwise).



Gear trains inside a mechanical watch

13.4. Compound Gear Train

When there are more than one gear on a shaft, as shown in Fig. 13.2, it is called a ***compound train of gear.***

We have seen in Art. 13.3 that the idle gears, in a simple train of gears do not effect the speed ratio of the system. But these gears are useful in bridging over the space between the driver and the driven.

But whenever the distance between the driver and the driven or follower has to be bridged over by intermediate gears and at the same time a great (or much less) speed ratio is required, then the advantage of intermediate gears is intensified by providing compound gears on intermediate shafts. In this case, each intermediate shaft has two gears rigidly fixed to it so that they may have the same speed. One of these two gears meshes with the driver and the other with the driven or follower attached to the next shaft as shown in Fig.13.2.

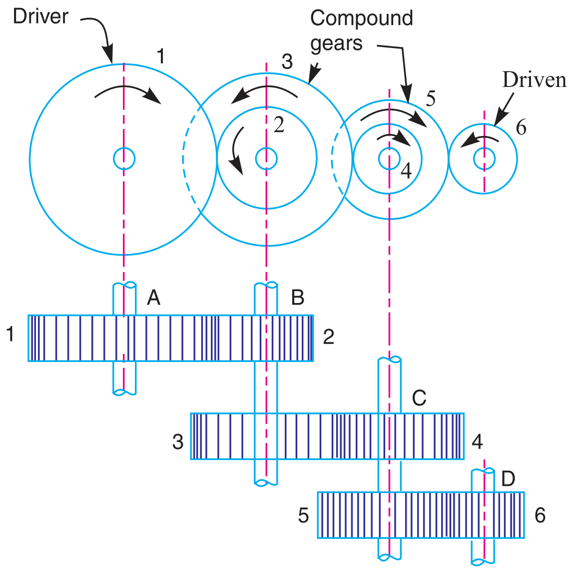


Fig. 13.2. Compound gear train.

In a compound train of gears, as shown in Fig. 13.2, the gear 1 is the driving gear mounted on shaft *A*, gears 2 and 3 are compound gears which are mounted on shaft *B*. The gears 4 and 5 are also compound gears which are mounted on shaft *C* and the gear 6 is the driven gear mounted on shaft *D*.

Let N_1 = Speed of driving gear 1,

T_1 = Number of teeth on driving gear 1,

$N_2, N_3 ..., N_6$ = Speed of respective gears in r.p.m., and

$T_2, T_3 ..., T_6$ = Number of teeth on respective gears.

Since gear 1 is in mesh with gear 2, therefore its speed ratio is

$$\frac{N_1}{N_2} = \frac{T_2}{T_1} \qquad ...(i)$$

Similarly, for gears 3 and 4, speed ratio is

$$\frac{N_3}{N_4} = \frac{T_4}{T_3} \qquad ...(ii)$$

and for gears 5 and 6, speed ratio is

$$\frac{N_5}{N_6} = \frac{T_6}{T_5} \qquad ...(iii)$$

The speed ratio of compound gear train is obtained by multiplying the equations (*i*), (*ii*) and (*iii*),

$$\therefore \quad \frac{N_1}{N_2} \times \frac{N_3}{N_4} \times \frac{N_5}{N_6} = \frac{T_2}{T_1} \times \frac{T_4}{T_3} \times \frac{T_6}{T_5} \quad \text{or} \quad {}^*\frac{N_1}{N_6} = \frac{T_2 \times T_4 \times T_6}{T_1 \times T_3 \times T_5}$$

* Since gears 2 and 3 are mounted on one shaft *B*, therefore $N_2 = N_3$. Similarly gears 4 and 5 are mounted on shaft *C*, therefore $N_4 = N_5$.

i.e.

$$\text{Speed ratio} = \frac{\text{Speed of the first driver}}{\text{Speed of the last driven or follower}}$$

$$= \frac{\text{Product of the number of teeth on the drivens}}{\text{Product of the number of teeth on the drivers}}$$

and

$$\text{Train value} = \frac{\text{Speed of the last driven or follower}}{\text{Speed of the first driver}}$$

$$= \frac{\text{Product of the number of teeth on the drivers}}{\text{Product of the number of teeth on the drivens}}$$

The advantage of a compound train over a simple gear train is that a much larger speed reduction from the first shaft to the last shaft can be obtained with small gears. If a simple gear train is used to give a large speed reduction, the last gear has to be very large. Usually for a speed reduction in excess of 7 to 1, a simple train is not used and a compound train or worm gearing is employed.

Note: The gears which mesh must have the same circular pitch or module. Thus gears 1 and 2 must have the same module as they mesh together. Similarly gears 3 and 4, and gears 5 and 6 must have the same module.

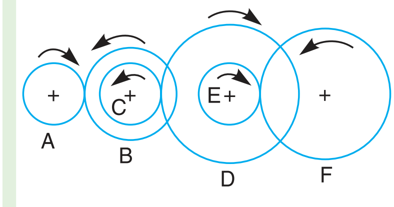


Fig. 13.3

Example 13.1. *The gearing of a machine tool is shown in Fig. 13.3. The motor shaft is connected to gear A and rotates at 975 r.p.m. The gear wheels B, C, D and E are fixed to parallel shafts rotating together. The final gear F is fixed on the output shaft. What is the speed of gear F ? The number of teeth on each gear are as given below :*

Gear	*A*	*B*	*C*	*D*	*E*	*F*
No. of teeth	*20*	*50*	*25*	*75*	*26*	*65*

Solution. Given : N_A = 975 r.p.m. ; $T_A = 20$; $T_B = 50$; $T_C = 25$; $T_D = 75$; $T_E = 26$; $T_F = 65$

From Fig. 13.3, we see that gears *A*, *C* and *E* are drivers while the gears *B*, *D* and *F* are driven or followers. Let the gear *A* rotates in clockwise direction. Since the gears *B* and *C* are mounted on the same shaft, therefore it is a compound gear and the direction or rotation of both these gears is same (*i.e.* anticlockwise). Similarly, the gears *D* and *E* are mounted on the same shaft, therefore it is also a compound gear and the direction of rotation of both these gears is same (*i.e.* clockwise). The gear *F* will rotate in anticlockwise direction.

Battery Car: Even though it is run by batteries, the power transmission, gears, clutches, brakes, etc. remain mechanical in nature.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Let N_F = Speed of gear *F*, *i.e.* last driven or follower.

We know that

$$\frac{\text{Speed of the first driver}}{\text{Speed of the last driven}} = \frac{\text{Product of no. of teeth on drivens}}{\text{Product of no. of teeth on drivers}}$$

or $$\frac{N_A}{N_F} = \frac{T_B \times T_D \times T_F}{T_A \times T_C \times T_E} = \frac{50 \times 75 \times 65}{20 \times 25 \times 26} = 18.75$$

$\therefore$ $$N_F = \frac{N_A}{18.75} = \frac{975}{18.75} = 52 \text{ r. p. m.}$$ **Ans.**

13.5. Design of Spur Gears

Sometimes, the spur gears (*i.e.* driver and driven) are to be designed for the given velocity ratio and distance between the centres of their shafts.

Let x = Distance between the centres of two shafts,

N_1 = Speed of the driver,

T_1 = Number of teeth on the driver,

d_1 = Pitch circle diameter of the driver,

N_2, T_2 and d_2 = Corresponding values for the driven or follower, and

p_c = Circular pitch.

We know that the distance between the centres of two shafts,

$$x = \frac{d_1 + d_2}{2} \qquad \text{...}(i)$$

and speed ratio or velocity ratio,

$$\frac{N_1}{N_2} = \frac{d_2}{d_1} = \frac{T_2}{T_1} \qquad \text{...}(ii)$$

From the above equations, we can conveniently find out the values of d_1 and d_2 (or T_1 and T_2) and the circular pitch (p_c). The values of T_1 and T_2, as obtained above, may or may not be whole numbers. But in a gear since the number of its teeth is always a whole number, therefore a slight alterations must be made in the values of x, d_1 and d_2, so that the number of teeth in the two gears may be a complete number.

Example 13.2. *Two parallel shafts, about 600 mm apart are to be connected by spur gears. One shaft is to run at 360 r.p.m. and the other at 120 r.p.m. Design the gears, if the circular pitch is to be 25 mm.*

Solution. Given : x = 600 mm ; N_1 = 360 r.p.m. ; N_2 = 120 r.p.m. ; p_c = 25 mm

Let d_1 = Pitch circle diameter of the first gear, and

d_2 = Pitch circle diameter of the second gear.

We know that speed ratio,

$$\frac{N_1}{N_2} = \frac{d_2}{d_1} = \frac{360}{120} = 3 \quad \text{or} \quad d_2 = 3d_1 \qquad \text{...}(i)$$

and centre distance between the shafts (x),

$$600 = \frac{1}{2}(d_1 + d_2) \quad \text{or} \quad d_1 + d_2 = 1200 \qquad \text{...}(ii)$$

From equations (*i*) and (*ii*), we find that

$$d_1 = 300 \text{ mm, and } d_2 = 900 \text{ mm}$$

$\therefore$ Number of teeth on the first gear,

$$T_1 = \frac{\pi d_2}{p_c} = \frac{\pi \times 300}{25} = 37.7$$

and number of teeth on the second gear,

$$T_2 = \frac{\pi d_2}{p_c} = \frac{\pi \times 900}{25} = 113.1$$

Since the number of teeth on both the gears are to be in complete numbers, therefore let us make the number of teeth on the first gear as 38. Therefore for a speed ratio of 3, the number of teeth on the second gear should be 38 × 3 = 114.

Now the exact pitch circle diameter of the first gear,

$$d_1' = \frac{T_1 \times p_c}{\pi} = \frac{38 \times 25}{\pi} = 302.36 \text{ mm}$$

and the exact pitch circle diameter of the second gear,

$$d_2' = \frac{T_2 \times p_c}{\pi} = \frac{114 \times 25}{\pi} = 907.1 \text{ mm}$$

∴ Exact distance between the two shafts,

$$x' = \frac{d_1' + d_2'}{2} = \frac{302.36 + 907.1}{2} = 604.73 \text{ mm}$$

Hence the number of teeth on the first and second gear must be 38 and 114 and their pitch circle diameters must be 302.36 mm and 907.1 mm respectively. The exact distance between the two shafts must be 604.73 mm. **Ans.**

13.6. Reverted Gear Train

When the axes of the first gear (*i.e.* first driver) and the last gear (*i.e.* last driven or follower) are co-axial, then the gear train is known as ***reverted gear train*** as shown in Fig. 13.4.

We see that gear 1 (*i.e.* first driver) drives the gear 2 (*i.e.* first driven or follower) in the opposite direction. Since the gears 2 and 3 are mounted on the same shaft, therefore they form a compound gear and the gear 3 will rotate in the same direction as that of gear 2. The gear 3 (which is now the second driver) drives the gear 4 (*i.e.* the last driven or follower) in the same direction as that of gear 1. Thus we see that in a reverted gear train, the motion of the first gear and the last gear is ***like.***

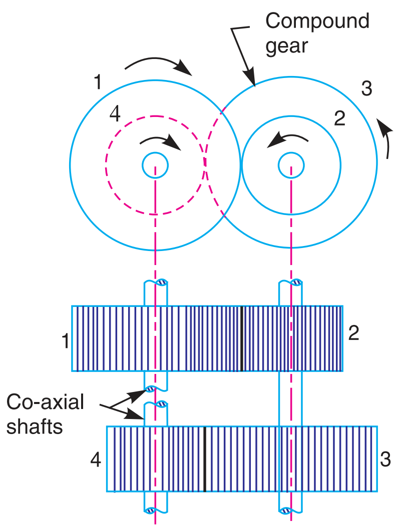


Fig. 13.4. Reverted gear train.

Let T_1 = Number of teeth on gear 1,

r_1 = Pitch circle radius of gear 1, and

N_1 = Speed of gear 1 in r.p.m.

Similarly,

T_2, T_3, T_4 = Number of teeth on respective gears,

r_2, r_3, r_4 = Pitch circle radii of respective gears, and

N_2, N_3, N_4 = Speed of respective gears in r.p.m.

Since the distance between the centres of the shafts of gears 1 and 2 as well as gears 3 and 4 is same, therefore

$$r_1 + r_2 = r_3 + r_4 \quad \text{...}(i)$$

Also, the circular pitch or module of all the gears is assumed to be same, therefore number of teeth on each gear is directly proportional to its circumference or radius.

$$\therefore \quad {}^{*}T_1 + T_2 = T_3 + T_4 \quad \text{...}(ii)$$

and

$$\text{Speed ratio} = \frac{\text{Product of number of teeth on drivens}}{\text{Product of number of teeth on drivers}}$$

or

$$\frac{N_1}{N_4} = \frac{T_2 \times T_4}{T_1 \times T_3} \quad \text{...}(iii)$$

From equations ***(i)***, ***(ii)*** and ***(iii)***, we can determine the number of teeth on each gear for the given centre distance, speed ratio and module only when the number of teeth on one gear is chosen arbitrarily.

The reverted gear trains are used in automotive transmissions, lathe back gears, industrial speed reducers, and in clocks (where the minute and hour hand shafts are co-axial).

Example 13.3. *The speed ratio of the reverted gear train, as shown in Fig. 13.5, is to be 12. The module pitch of gears A and B is 3.125 mm and of gears C and D is 2.5 mm. Calculate the suitable numbers of teeth for the gears. No gear is to have less than 24 teeth.*

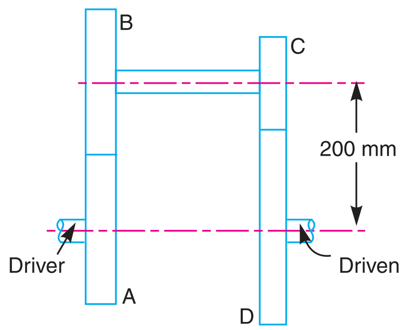


Fig. 13.5

Solution. Given : Speed ratio, $N_A/N_D = 12$; $m_A = m_B = 3.125$ mm ; $m_C = m_D = 2.5$ mm

Let N_A = Speed of gear A,

T_A = Number of teeth on gear A,

r_A = Pitch circle radius of gear A,

N_B, N_C, N_D = Speed of respective gears,

T_B, T_C, T_D = Number of teeth on respective gears, and

r_B, r_C, r_D = Pitch circle radii of respective gears.

---

* We know that circular pitch,

$$p_c = \frac{2\pi r}{T} = \pi m \quad \text{or} \quad r = \frac{m.T}{2}, \text{ where } m \text{ is the module.}$$

$$\therefore \quad r_1 = \frac{m.T_1}{2} \; ; \; r_2 = \frac{m.T_2}{2} \; ; \; r_3 = \frac{m.T_3}{2} \; ; \; r_4 = \frac{m.T_4}{2}$$

Now from equation ***(i)***,

$$\frac{m.T_1}{2} + \frac{m.T_2}{2} = \frac{m.T_3}{2} + \frac{m.T_4}{2}$$

$$T_1 + T_2 = T_3 + T_4$$

Since the speed ratio between the gears *A* and *B* and between the gears *C* and *D* are to be same, therefore

$$^*\frac{N_A}{N_B} = \frac{N_C}{N_D} = \sqrt{12} = 3.464$$

Also the speed ratio of any pair of gears in mesh is the inverse of their number of teeth, therefore

$$\frac{T_B}{T_A} = \frac{T_D}{T_C} = 3.464 \qquad \text{...}(i)$$

We know that the distance between the shafts

$$x = r_A + r_B = r_C + r_D = 200 \text{ mm}$$

or $$\frac{m_A.T_A}{2} + \frac{m_B.T_B}{2} = \frac{m_C.T_C}{2} + \frac{m_D.T_D}{2} = 200 \qquad \text{...}\left(\because r = \frac{m.T}{2}\right)$$

$$3.125\,(T_A + T_B) = 2.5\,(T_C + T_D) = 400 \qquad \text{...}(\because m_A = m_B, \text{ and } m_C = m_D)$$

$\therefore$ $$T_A + T_B = 400 / 3.125 = 128 \qquad \text{...}(ii)$$

and $$T_C + T_D = 400 / 2.5 = 160 \qquad \text{...}(iii)$$

From equation (*i*), $T_B = 3.464\ T_A$. Substituting this value of T_B in equation (*ii*),

$$T_A + 3.464\ T_A = 128 \quad \text{or} \quad T_A = 128 / 4.464 = 28.67 \text{ say } 28 \ \textbf{Ans.}$$

and $$T_B = 128 - 28 = 100 \ \textbf{Ans.}$$

Again from equation (*i*), $T_D = 3.464\ T_C$. Substituting this value of T_D in equation (*iii*),

$$T_C + 3.464\ T_C = 160 \quad \text{or} \quad T_C = 160 / 4.464 = 35.84 \text{ say } 36 \ \textbf{Ans.}$$

and $$T_D = 160 - 36 = 124 \ \textbf{Ans.}$$

Note : The speed ratio of the reverted gear train with the calculated values of number of teeth on each gear is

$$\frac{N_A}{N_D} = \frac{T_B \times T_D}{T_A \times T_C} = \frac{100 \times 124}{28 \times 36} = 12.3$$

13.7. Epicyclic Gear Train

We have already discussed that in an epicyclic gear train, the axes of the shafts, over which the gears are mounted, may move relative to a fixed axis. A simple epicyclic gear train is shown in Fig. 13.6, where a gear *A* and the arm *C* have a common axis at O_1 about which they can rotate. The gear *B* meshes with gear *A* and has its axis on the arm at O_2, about which the gear *B* can rotate. If the

---

* We know that speed ratio $= \dfrac{\text{Speed of first driver}}{\text{Speed of last driven}} = \dfrac{N_A}{N_D} = 12$

Also $$\frac{N_A}{N_D} = \frac{N_A}{N_B} \times \frac{N_C}{N_D} \qquad \text{...}(N_B = N_C, \text{ being on the same shaft})$$

For $\dfrac{N_A}{N_B}$ and $\dfrac{N_C}{N_D}$ to be same, each speed ratio should be $\sqrt{12}$ so that

$$\frac{N_A}{N_D} = \frac{N_A}{N_B} \times \frac{N_C}{N_D} = \sqrt{12} \times \sqrt{12} = 12$$

arm is fixed, the gear train is simple and gear A can drive gear B or ***vice- versa***, but if gear A is fixed and the arm is rotated about the axis of gear A (*i.e.* O_1), then the gear B is forced to rotate ***upon*** and ***around*** gear A. Such a motion is called **epicyclic** and the gear trains arranged in such a manner that one or more of their members move upon and around another member are known as ***epicyclic gear trains*** (***epi.*** means upon and ***cyclic*** means around). The epicyclic gear trains may be ***simple*** or ***compound***.

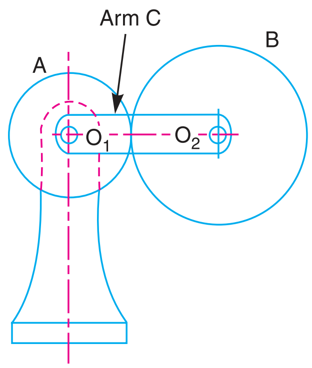


Fig. 13.6. Epicyclic gear train.

The epicyclic gear trains are useful for transmitting high velocity ratios with gears of moderate size in a comparatively lesser space. The epicyclic gear trains are used in the back gear of lathe, differential gears of the automobiles, hoists, pulley blocks, wrist watches etc.

13.8. Velocity Ratioz of Epicyclic Gear Train

The following two methods may be used for finding out the velocity ratio of an epicyclic gear train.

1. Tabular method, and **2.** Algebraic method.

These methods are discussed, in detail, as follows :

1. *Tabular method.* Consider an epicyclic gear train as shown in Fig. 13.6.

Let T_A = Number of teeth on gear A, and

T_B = Number of teeth on gear B.

First of all, let us suppose that the arm is fixed. Therefore the axes of both the gears are also fixed relative to each other. When the gear A makes one revolution anticlockwise, the gear B will make *T_A / T_B revolutions, clockwise. Assuming the anticlockwise rotation as positive and clockwise as negative, we may say that when gear A makes + 1 revolution, then the gear B will make $(– T_A / T_B)$ revolutions. This statement of relative motion is entered in the first row of the table (see Table 13.1).

Inside view of a car engine.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Secondly, if the gear A makes $+ x$ revolutions, then the gear B will make $– x \times T_A / T_B$ revolutions. This statement is entered in the second row of the table. In other words, multiply the each motion (entered in the first row) by x.

Thirdly, each element of an epicyclic train is given $+ y$ revolutions and entered in the third row. Finally, the motion of each element of the gear train is added up and entered in the fourth row.

* We know that $N_B / N_A = T_A / T_B$. Since N_A = 1 revolution, therefore $N_B = T_A / T_B$.

Table 13.1. Table of motions

Step No.	Conditions of motion	Revolutions of elements		
		Arm C	*Gear A*	*Gear B*
1.	Arm fixed-gear *A* rotates through + 1 revolution *i.e.* 1 rev. anticlockwise	0	+ 1	$-\dfrac{T_A}{T_B}$
2.	Arm fixed-gear *A* rotates through + *x* revolutions	0	+ *x*	$-x \times \dfrac{T_A}{T_B}$
3.	Add + *y* revolutions to all elements	+ *y*	+ *y*	+ *y*
4.	Total motion	+ *y*	*x* + *y*	$y - x \times \dfrac{T_A}{T_B}$

A little consideration will show that when two conditions about the motion of rotation of any two elements are known, then the unknown speed of the third element may be obtained by substituting the given data in the third column of the fourth row.

2. ***Algebraic method.*** In this method, the motion of each element of the epicyclic train relative to the arm is set down in the form of equations. The number of equations depends upon the number of elements in the gear train. But the two conditions are, usually, supplied in any epicyclic train *viz.* some element is fixed and the other has specified motion. These two conditions are sufficient to solve all the equations ; and hence to determine the motion of any element in the epicyclic gear train.

Let the arm *C* be fixed in an epicyclic gear train as shown in Fig. 13.6. Therefore speed of the gear *A* relative to the arm *C*

$$= N_A - N_C$$

and speed of the gear *B* relative to the arm *C*,

$$= N_B - N_C$$

Since the gears *A* and *B* are meshing directly, therefore they will revolve in ***opposite*** directions.

$$\therefore \qquad \frac{N_B - N_C}{N_A - N_C} = -\frac{T_A}{T_B}$$

Since the arm *C* is fixed, therefore its speed, $N_C = 0$.

$$\therefore \qquad \frac{N_B}{N_A} = -\frac{T_A}{T_B}$$

If the gear *A* is fixed, then $N_A = 0$.

$$\frac{N_B - N_C}{0 - N_C} = -\frac{T_A}{T_B} \quad \text{or} \quad \frac{N_B}{N_C} = 1 + \frac{T_A}{T_B}$$

Note : The tabular method is easier and hence mostly used in solving problems on epicyclic gear train.

Example 13.4. *In an epicyclic gear train, an arm carries two gears A and B having 36 and 45 teeth respectively. If the arm rotates at 150 r.p.m. in the anticlockwise direction about the centre of the gear A which is fixed, determine the speed of gear B. If the gear A instead of being fixed, makes 300 r.p.m. in the clockwise direction, what will be the speed of gear B ?*

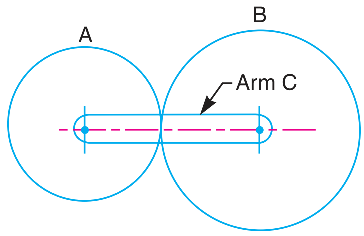


Fig. 13.7

Solution. Given : $T_A = 36$; $T_B = 45$; $N_C = 150$ r.p.m. (anticlockwise)

The gear train is shown in Fig. 13.7.

We shall solve this example, first by tabular method and then by algebraic method.

1. *Tabular method*

First of all prepare the table of motions as given below :

Table 13.2. Table of motions.

Step No.	Conditions of motion	Revolutions of elements		
		Arm C	Gear A	Gear B
1.	Arm fixed-gear A rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\dfrac{T_A}{T_B}$
2.	Arm fixed-gear A rotates through + x revolutions	0	$+x$	$-x \times \dfrac{T_A}{T_B}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \times \dfrac{T_A}{T_B}$

Speed of gear B when gear A is fixed

Since the speed of arm is 150 r.p.m. anticlockwise, therefore from the fourth row of the table,

$$y = +150 \text{ r.p.m.}$$

Also the gear A is fixed, therefore

$$x + y = 0 \quad \text{or} \quad x = -y = -150 \text{ r.p.m.}$$

$\therefore$ Speed of gear B, $N_B = y - x \times \dfrac{T_A}{T_B} = 150 + 150 \times \dfrac{36}{45} = +270$ r.p.m.

$$= 270 \text{ r.p.m. (anticlockwise)}$$ **Ans.**

Speed of gear B when gear A makes 300 r.p.m. clockwise

Since the gear A makes 300 r.p.m.clockwise, therefore from the fourth row of the table,

$$x + y = -300 \quad \text{or} \quad x = -300 - y = -300 - 150 = -450 \text{ r.p.m.}$$

$\therefore$ Speed of gear B,

$$N_B = y - x \times \frac{T_A}{T_B} = 150 + 450 \times \frac{36}{45} = +510 \text{ r.p.m.}$$

$$= 510 \text{ r.p.m. (anticlockwise)}$$ **Ans.**

2. *Algebraic method*

Let N_A = Speed of gear A.

N_B = Speed of gear B, and

N_C = Speed of arm C.

Assuming the arm C to be fixed, speed of gear A relative to arm C

$$= N_A - N_C$$

and speed of gear B relative to arm $C = N_B - N_C$

Since the gears A and B revolve in ***opposite*** directions, therefore

$$\frac{N_B - N_C}{N_A - N_C} = -\frac{T_A}{T_B} \qquad \text{...}(i)$$

Speed of gear B when gear A is fixed

When gear A is fixed, the arm rotates at 150 r.p.m. in the anticlockwise direction, *i.e.*

$$N_A = 0, \quad \text{and} \quad N_C = +\,150 \text{ r.p.m.}$$

$$\therefore \quad \frac{N_B - 150}{0 - 150} = -\frac{36}{45} = -\,0.8 \qquad \text{...[From equation } (i)]$$

or $$N_B = -\,150 \times -\,0.8 + 150 = 120 + 150 = 270 \text{ r.p.m.}$$ **Ans.**

Speed of gear B when gear A makes 300 r.p.m. clockwise

Since the gear A makes 300 r.p.m. clockwise, therefore

$$N_A = -\,300 \text{ r.p.m.}$$

$$\therefore \quad \frac{N_B - 150}{-300 - 150} = -\frac{36}{45} = -\,0.8$$

or $$N_B = -\,450 \times -\,0.8 + 150 = 360 + 150 = 510 \text{ r.p.m.}$$ **Ans.**

Example 13.5. *In a reverted epicyclic gear train, the arm A carries two gears B and C and a compound gear D - E. The gear B meshes with gear E and the gear C meshes with gear D. The number of teeth on gears B, C and D are 75, 30 and 90 respectively. Find the speed and direction of gear C when gear B is fixed and the arm A makes 100 r.p.m. clockwise.*

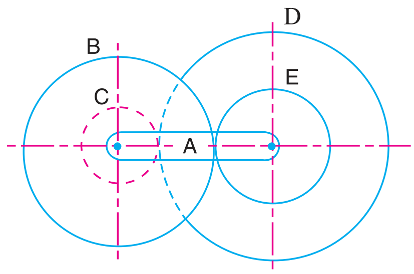


Fig. 13.8

Solution. Given : $T_B = 75$; $T_C = 30$; $T_D = 90$; $N_A = 100$ r.p.m. (clockwise)

The reverted epicyclic gear train is shown in Fig. 13.8. First of all, let us find the number of teeth on gear E (T_E). Let d_B, d_C, d_D and d_E be the pitch circle diameters of gears B, C, D and E respectively. From the geometry of the figure,

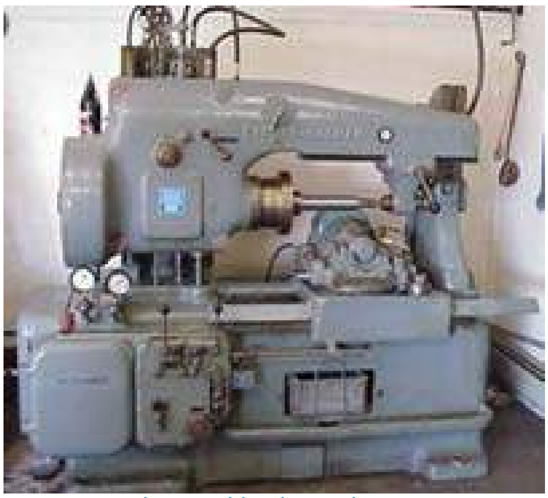
$$d_B + d_E = d_C + d_D$$

Since the number of teeth on each gear, for the same module, are proportional to their pitch circle diameters, therefore

$$T_B + T_E = T_C + T_D$$

$$\therefore \quad T_E = T_C + T_D - T_B = 30 + 90 - 75 = 45$$

The table of motions is drawn as follows :



A gear-cutting machine is used to cut gears.

Note : This picture is given as additional information and is not a direct example of the current chapter.

Table 13.3. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Arm A	Compound gear D-E	Gear B	Gear C
1.	Arm fixed-compound gear *D-E* rotated through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\frac{T_E}{T_B}$	$-\frac{T_D}{T_C}$
2.	Arm fixed-compound gear *D-E* rotated through + *x* revolutions	0	$+x$	$-x \times \frac{T_E}{T_B}$	$-x \times \frac{T_D}{T_C}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y-x \times \frac{T_E}{T_B}$	$y-x \times \frac{T_D}{T_C}$

Since the gear *B* is fixed, therefore from the fourth row of the table,

$$y - x \times \frac{T_E}{T_B} = 0 \quad \text{or} \quad y - x \times \frac{45}{75} = 0$$

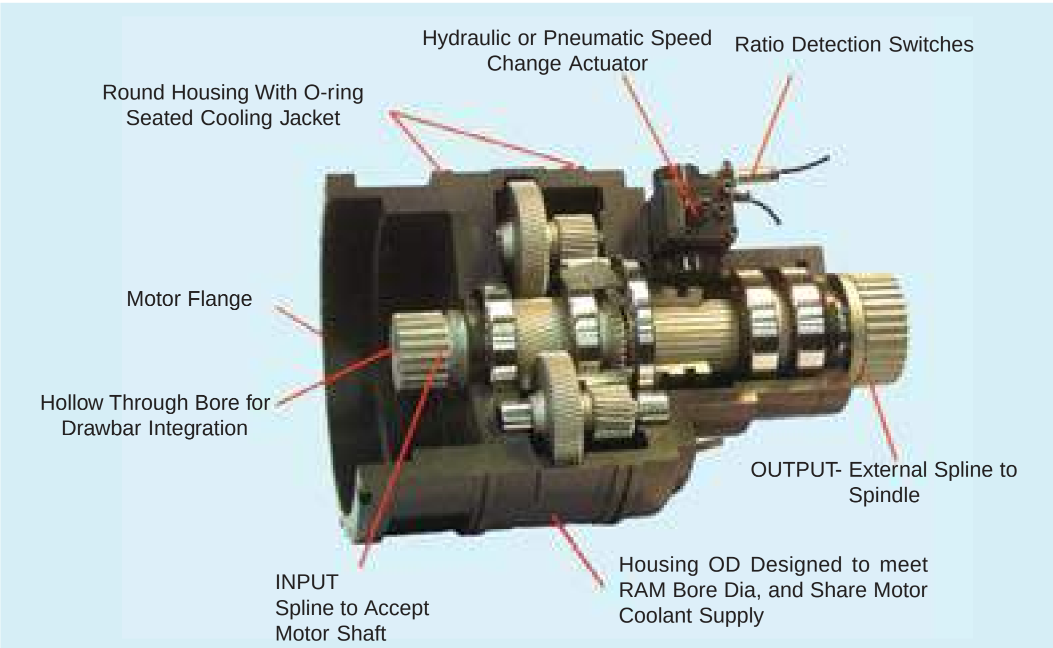
$$\therefore \qquad y - 0.6 = 0 \qquad \text{...(i)}$$

Also the arm *A* makes 100 r.p.m. clockwise, therefore

$$y = -100 \qquad \text{...(ii)}$$

Substituting $y = -100$ in equation *(i)*, we get

$$-100 - 0.6\,x = 0 \quad \text{or} \quad x = -100 / 0.6 = -166.67$$



Model of sun and planet gears.

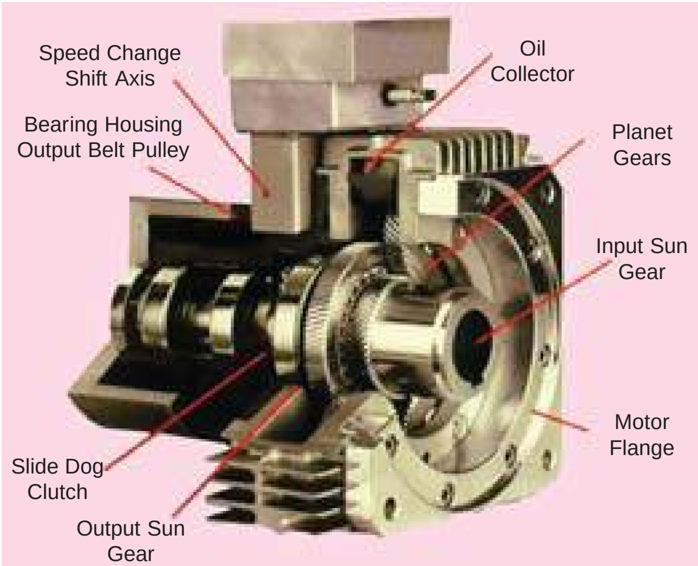
From the fourth row of the table, speed of gear C,

$$N_C = y - x \times \frac{T_D}{T_C} = -100 + 166.67 \times \frac{90}{30} = +400 \text{ r.p.m.}$$

$$= 400 \text{ r.p.m. (anticlockwise)} \textbf{ Ans.}$$

13.9. Compound Epicyclic Gear Train—Sun and Planet Gear

A compound epicyclic gear train is shown in Fig. 13.9. It consists of two co-axial shafts S_1 and S_2, an annulus gear A which is fixed, the compound gear (or planet gear) B-C, the sun gear D and the arm H. The annulus gear has internal teeth and the compound gear is carried by the arm and revolves freely on a pin of the arm H. The sun gear is co-axial with the annulus gear and the arm but independent of them.



Sun and Planet gears.

The annulus gear A meshes with the gear B and the sun gear D meshes with the gear C. It may be noted that when the annulus gear is fixed, the sun gear provides the drive and when the sun gear is fixed, the annulus gear provides the drive. In both cases, the arm acts as a follower.

Note : The gear at the centre is called the ***sun gear*** and the gears whose axes move are called ***planet gears***.

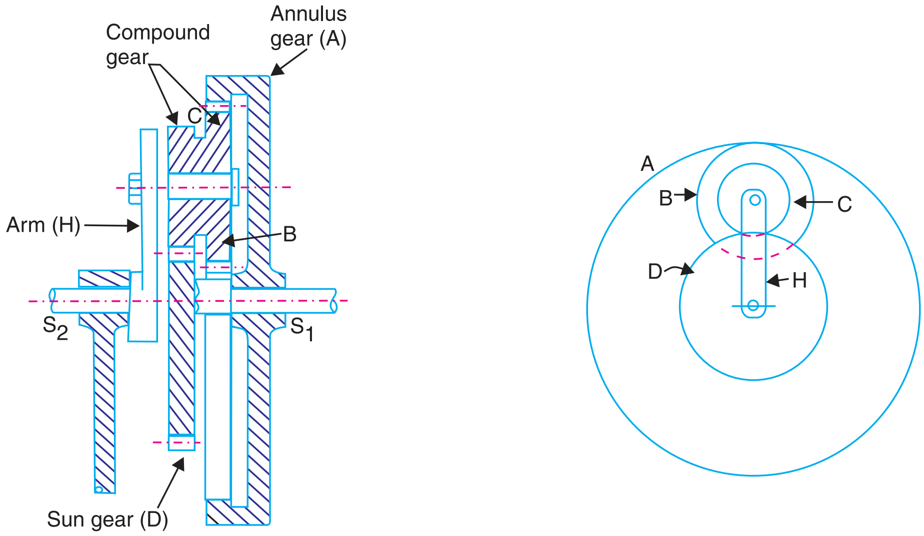


Fig. 13.9. Compound epicyclic gear train.

Let T_A, T_B, T_C, and T_D be the teeth and N_A, N_B, N_C and N_D be the speeds for the gears *A*, *B*, *C* and *D* respectively. A little consideration will show that when the arm is fixed and the sun gear *D* is turned anticlockwise, then the compound gear *B*-*C* and the annulus gear A will rotate in the clockwise direction.

The motion of rotations of the various elements are shown in the table below.

Table 13.4. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Arm	*Gear D*	*Compound gear B-C*	*Gear A*
1.	Arm fixed-gear *D* rotates through + 1 revolution	0	+ 1	$-\frac{T_D}{T_C}$	$-\frac{T_D}{T_C} \times \frac{T_B}{T_A}$
2.	Arm fixed-gear *D* rotates through + *x* revolutions	0	$+x$	$-x \times \frac{T_D}{T_C}$	$-x \times \frac{T_D}{T_C} \times \frac{T_B}{T_A}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \times \frac{T_D}{T_C}$	$y - x \times \frac{T_D}{T_C} \times \frac{T_B}{T_A}$

Note : If the annulus gear *A* is rotated through one revolution anticlockwise with the arm fixed, then the compound gear rotates through T_A / T_B revolutions in the same sense and the sun gear *D* rotates through $T_A / T_B \times T_C / T_D$ revolutions in clockwise direction.

Example 13.6. *An epicyclic gear consists of three gears A, B and C as shown in Fig. 13.10. The gear A has 72 internal teeth and gear C has 32 external teeth. The gear B meshes with both A and C and is carried on an arm EF which rotates about the centre of A at 18 r.p.m.. If the gear A is fixed, determine the speed of gears B and C.*

Solution. Given : $T_A = 72$; $T_C = 32$; Speed of arm EF = 18 r.p.m.

Considering the relative motion of rotation as shown in Table 13.5.

Table 13.5. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Arm EF	*Gear C*	*Gear B*	*Gear A*
1.	Arm fixed-gear *C* rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\frac{T_C}{T_B}$	$-\frac{T_C}{T_B} \times \frac{T_B}{T_A} = -\frac{T_C}{T_A}$
2.	Arm fixed-gear *C* rotates through + *x* revolutions	0	$+x$	$-x \times \frac{T_C}{T_B}$	$-x \times \frac{T_C}{T_A}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \times \frac{T_C}{T_B}$	$y - x \times \frac{T_C}{T_A}$

Speed of gear C

We know that the speed of the arm is 18 r.p.m. therefore,

$$y = 18 \text{ r.p.m.}$$

and the gear A is fixed, therefore

$$y - x \times \frac{T_C}{T_A} = 0 \quad \text{or} \quad 18 - x \times \frac{32}{72} = 0$$

$$\therefore \quad x = 18 \times 72 / 32 = 40.5$$

$\therefore$ Speed of gear C $= x + y = 40.5 + 18$

$= + 58.5$ r.p.m.

$= 58.5$ r.p.m. in the direction of arm. **Ans.**

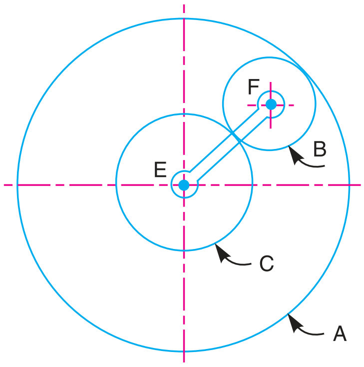


Fig. 13.10

Speed of gear B

Let d_A, d_B and d_C be the pitch circle diameters of gears A, B and C respectively. Therefore, from the geometry of Fig. 13.10,

$$d_B + \frac{d_C}{2} = \frac{d_A}{2} \quad \text{or} \quad 2\,d_B + d_C = d_A$$

Since the number of teeth are proportional to their pitch circle diameters, therefore

$$2\,T_B + T_C = T_A \quad \text{or} \quad 2\,T_B + 32 = 72 \quad \text{or} \quad T_B = 20$$

$\therefore$ Speed of gear B $= y - x \times \frac{T_C}{T_B} = 18 - 40.5 \times \frac{32}{20} = -46.8$ r.p.m.

$= 46.8$ r.p.m. in the opposite direction of arm. **Ans.**

Example 13.7. *An epicyclic train of gears is arranged as shown in Fig.13.11. How many revolutions does the arm, to which the pinions B and C are attached, make :*

1. *when A makes one revolution clockwise and D makes half a revolution anticlockwise, and*

2. *when A makes one revolution clockwise and D is stationary ?*

The number of teeth on the gears A and D are 40 and 90 respectively.

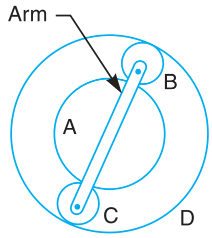


Fig. 13.11

Solution. Given : $T_A = 40$; $T_D = 90$

First of all, let us find the number of teeth on gears B and C (*i.e.* T_B and T_C). Let d_A, d_B, d_C and d_D be the pitch circle diameters of gears A, B, C and D respectively. Therefore from the geometry of the figure,

$$d_A + d_B + d_C = d_D \quad \text{or} \quad d_A + 2\,d_B = d_D \qquad ...(\because\ d_B = d_C)$$

Since the number of teeth are proportional to their pitch circle diameters, therefore,

$$T_A + 2\,T_B = T_D \quad \text{or} \quad 40 + 2\,T_B = 90$$

$$\therefore \quad T_B = 25, \quad \text{and} \quad T_C = 25 \qquad ...(\because\ T_B = T_C)$$

The table of motions is given below :

Table 13.6. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Arm	Gear A	Compound gear B-C	Gear D
1.	Arm fixed, gear *A* rotates through – 1 revolution (*i.e.* 1 rev. clockwise)	0	– 1	$+\frac{T_A}{T_B}$	$+\frac{T_A}{T_B}\times\frac{T_B}{T_D}=+\frac{T_A}{T_D}$
2.	Arm fixed, gear *A* rotates through – *x* revolutions	0	$-x$	$+x\times\frac{T_A}{T_B}$	$+x\times\frac{T_A}{T_D}$
3.	Add – *y* revolutions to all elements	$-y$	$-y$	$-y$	$-y$
4.	Total motion	$-y$	$-x-y$	$x\times\frac{T_A}{T_B}-y$	$x\times\frac{T_A}{T_D}-y$

1. Speed of arm when A makes 1 revolution clockwise and D makes half revolution anticlockwise

Since the gear *A* makes 1 revolution clockwise, therefore from the fourth row of the table,

$$-x-y=-1 \quad \text{or} \quad x+y=1 \qquad \ldots(i)$$

Also, the gear *D* makes half revolution anticlockwise, therefore

$$x\times\frac{T_A}{T_D}-y=\frac{1}{2} \quad \text{or} \quad x\times\frac{40}{90}-y=\frac{1}{2}$$

$$\therefore \quad 40\,x-90\,y=45 \quad \text{or} \quad x-2.25\,y=1.125 \qquad \ldots(ii)$$

From equations (*i*) and (*ii*), $x = 1.04$ and $y = -0.04$

$$\therefore \quad \text{Speed of arm} = -y = -(-0.04) = +0.04$$

$$= 0.04 \text{ revolution anticlockwise} \quad \textbf{Ans.}$$

2. Speed of arm when A makes 1 revolution clockwise and D is stationary

Since the gear *A* makes 1 revolution clockwise, therefore from the fourth row of the table,

$$-x-y=-1 \quad \text{or} \quad x+y=1 \qquad \ldots(iii)$$

Also the gear *D* is stationary, therefore

$$x\times\frac{T_A}{T_D}-y=0 \quad \text{or} \quad x\times\frac{40}{90}-y=0$$

$$\therefore \quad 40\,x-90\,y=0 \quad \text{or} \quad x-2.25\,y=0 \qquad \ldots(iv)$$

From equations (*iii*) and (*iv*),

$$x = 0.692 \quad \text{and} \quad y = 0.308$$

$\therefore$ Speed of arm $= -y = -0.308 = 0.308$ revolution clockwise **Ans.**

Example 13.8. *In an epicyclic gear train, the internal wheels A and B and compound wheels C and D rotate independently about axis O. The wheels E and F rotate on pins fixed to the arm G. E gears with A and C and F gears with B and D. All the wheels have the same module and the number of teeth are :* $T_C = 28;\ T_D = 26;\ T_E = T_F = 18.$

1. Sketch the arrangement ; 2. Find the number of teeth on A and B ; 3. If the arm G makes 100 r.p.m. clockwise and A is fixed, find the speed of B ; and 4. If the arm G makes 100 r.p.m. clockwise and wheel A makes 10 r.p.m. counter clockwise ; find the speed of wheel B.

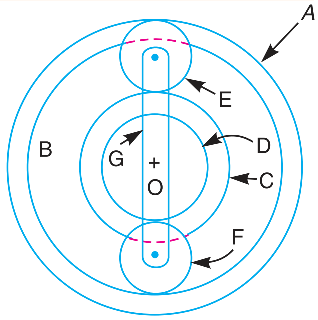


Fig. 13.12

Solution. Given : $T_C = 28$; $T_D = 26$; $T_E = T_F = 18$

1. Sketch the arrangement

The arrangement is shown in Fig. 13.12.

2. Number of teeth on wheels A and B

Let T_A = Number of teeth on wheel A, and

T_B = Number of teeth on wheel B.

If d_A , d_B , d_C , d_D , d_E and d_F are the pitch circle diameters of wheels A, B, C, D, E and F respectively, then from the geometry of Fig. 13.12,

$$d_A = d_C + 2\, d_E$$

and $$d_B = d_D + 2\, d_F$$

Since the number of teeth are proportional to their pitch circle diameters, for the same module, therefore

$$T_A = T_C + 2\,T_E = 28 + 2 \times 18 = 64 \quad \textbf{Ans.}$$

and $$T_B = T_D + 2\,T_F = 26 + 2 \times 18 = 62 \quad \textbf{Ans.}$$

3. Speed of wheel B when arm G makes 100 r.p.m. clockwise and wheel A is fixed

First of all, the table of motions is drawn as given below :

Table 13.7. Table of motions.

Step No.	Conditions of motion	Revolutions of elements					
		Arm G	Wheel A	Wheel E	Compound wheel C-D	Wheel F	Wheel B
1.	Arm fixed- wheel *A* rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$+\frac{T_A}{T_E}$	$-\frac{T_A}{T_E}\times\frac{T_E}{T_C}$ $= -\frac{T_A}{T_C}$	$+\frac{T_A}{T_C}\times\frac{T_D}{T_F}$	$+\frac{T_A}{T_C}\times\frac{T_D}{T_F}\times\frac{T_F}{T_B}$ $= +\frac{T_A}{T_C}\times\frac{T_D}{T_B}$
2.	Arm fixed-wheel *A* rotates through + *x* revolutions	0	$+x$	$+x\times\frac{T_A}{T_E}$	$-x\times\frac{T_A}{T_C}$	$+x\times\frac{T_A}{T_C}\times\frac{T_D}{T_F}$	$+x\times\frac{T_A}{T_C}\times\frac{T_D}{T_B}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y+x\times\frac{T_A}{T_E}$	$y-x\times\frac{T_A}{T_C}$	$y+x\times\frac{T_A}{T_C}\times\frac{T_D}{T_F}$	$y+x\times\frac{T_A}{T_C}\times\frac{T_D}{T_B}$

Since the arm *G* makes 100 r.p.m. clockwise, therefore from the fourth row of the table,

$$y = -100 \quad \text{...(i)}$$

Also, the wheel *A* is fixed, therefore from the fourth row of the table,

$$x + y = 0 \quad \text{or} \quad x = -y = 100 \quad \text{...(ii)}$$

$$\therefore \quad \text{Speed of wheel } B = y + x \times \frac{T_A}{T_C} \times \frac{T_D}{T_B} = -100 + 100 \times \frac{64}{28} \times \frac{26}{62} = -100 + 95.8 \text{ r.p.m.}$$

$$= -4.2 \text{ r.p.m.} = 4.2 \text{ r.p.m. clockwise} \quad \textbf{Ans.}$$

4. *Speed of wheel B when arm G makes 100 r.p.m. clockwise and wheel A makes 10 r.p.m. counter clockwise*

Since the arm *G* makes 100 r.p.m. clockwise, therefore from the fourth row of the table

$$y = -100 \quad \text{...(iii)}$$

Also the wheel *A* makes 10 r.p.m. counter clockwise, therefore from the fourth row of the table,

$$x + y = 10 \quad \text{or} \quad x = 10 - y = 10 + 100 = 110 \quad \text{...(iv)}$$

$$\therefore \quad \text{Speed of wheel } B = y + x \times \frac{T_A}{T_C} \times \frac{T_D}{T_B} = -100 + 110 \times \frac{64}{28} \times \frac{26}{62} = -100 + 105.4 \text{ r.p.m.}$$

$$= +5.4 \text{ r.p.m.} = 5.4 \text{ r.p.m. counter clockwise} \quad \textbf{Ans.}$$

Example 13.9. *In an epicyclic gear of the 'sun and planet' type shown in Fig. 13.13, the pitch circle diameter of the internally toothed ring is to be 224 mm and the module 4 mm. When the ring D is stationary, the spider A, which carries three planet wheels C of equal size, is to make one revolution in the same sense as the sunwheel B for every five revolutions of the driving spindle carrying the sunwheel B. Determine suitable numbers of teeth for all the wheels.*

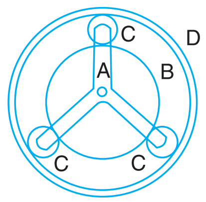


Fig. 13.13

Solution. Given : $d_D = 224$ mm ; $m = 4$ mm ; $N_A = N_B / 5$

Let T_B, T_C and T_D be the number of teeth on the sun wheel *B*, planet wheels *C* and the internally toothed ring *D*. The table of motions is given below :

Table 13.8. Table of motions.

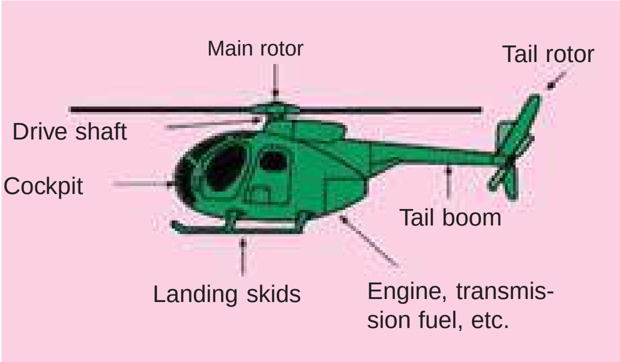
Step No.	Conditions of motion	Revolutions of elements: Spider A	Sun wheel B	Planet wheel C	Internal gear D
1.	Spider *A* fixed, sun wheel *B* rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\frac{T_B}{T_C}$	$-\frac{T_B}{T_C} \times \frac{T_C}{T_D} = -\frac{T_B}{T_D}$
2.	Spider *A* fixed, sun wheel *B* rotates through + *x* revolutions	0	+ *x*	$-x \times \frac{T_B}{T_C}$	$-x \times \frac{T_B}{T_D}$
3.	Add + *y* revolutions to all elements	+ *y*	+ *y*	+ *y*	+ *y*
4.	Total motion	+ *y*	*x* + *y*	$y - x \times \frac{T_B}{T_C}$	$y - x \times \frac{T_B}{T_D}$

We know that when the sun wheel B makes + 5 revolutions, the spider A makes + 1 revolution. Therefore from the fourth row of the table,

$$y = +1 \; ; \quad \text{and} \quad x + y = +5$$

$$\therefore \quad x = 5 - y = 5 - 1 = 4$$

Since the internally toothed ring D is stationary, therefore from the fourth row of the table,



Power transmission in a helicopter is essentially through gear trains.

Note : This picture is given as additional information and is not a direct example of the current chapter.

$$y - x \times \frac{T_B}{T_D} = 0$$

or
$$1 - 4 \times \frac{T_B}{T_D} = 0$$

$$\therefore \quad \frac{T_B}{T_D} = \frac{1}{4} \quad \text{or} \quad T_D = 4\,T_B \qquad \ldots(i)$$

We know that $\quad T_D = d_D / m = 224 / 4 = 56$ **Ans.**

$\therefore \quad T_B = T_D / 4 = 56 / 4 = 14$ **Ans.** ...[From equation (*i*)]

Let d_B, d_C and d_D be the pitch circle diameters of sun wheel B, planet wheels C and internally toothed ring D respectively. Assuming the pitch of all the gears to be same, therefore from the geometry of Fig. 13.13,

$$d_B + 2\,d_C = d_D$$

Since the number of teeth are proportional to their pitch circle diameters, therefore

$$T_B + 2\,T_C = T_D \quad \text{or} \quad 14 + 2\,T_C = 56$$

$\therefore \quad T_C = 21$ **Ans.**

Example 13.10. *Two shafts A and B are co-axial. A gear C (50 teeth) is rigidly mounted on shaft A. A compound gear D-E gears with C and an internal gear G. D has 20 teeth and gears with C and E has 35 teeth and gears with an internal gear G. The gear G is fixed and is concentric with the shaft axis. The compound gear D-E is mounted on a pin which projects from an arm keyed to the shaft B. Sketch the arrangement and find the number of teeth on internal gear G assuming that all gears have the same module. If the shaft A rotates at 110 r.p.m., find the speed of shaft B.*

Solution. Given : $T_C = 50$; $T_D = 20$; $T_E = 35$; $N_A = 110$ r.p.m.

The arrangement is shown in Fig. 13.14.

Number of teeth on internal gear G

Let d_C, d_D, d_E and d_G be the pitch circle diameters of gears C, D, E and G respectively. From the geometry of the figure,

$$\frac{d_G}{2} = \frac{d_C}{2} + \frac{d_D}{2} + \frac{d_E}{2}$$

or
$$d_G = d_C + d_D + d_E$$

Let T_C, T_D, T_E and T_G be the number of teeth on gears *C, D, E* and *G* respectively. Since all the gears have the same module, therefore number of teeth are proportional to their pitch circle diameters.

$\therefore$ $$T_G = T_C + T_D + T_E = 50 + 20 + 35 = 105 \text{ Ans.}$$

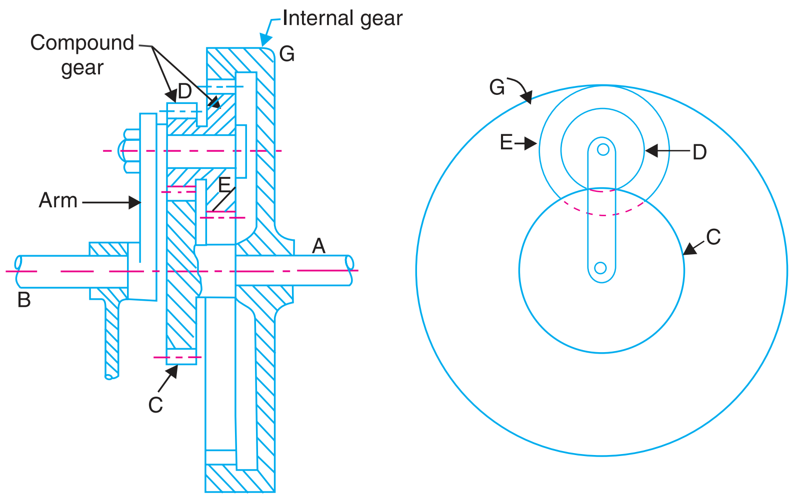


Fig. 13.14

Speed of shaft B

The table of motions is given below :

Table 13.9. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Arm	*Gear C (or shaft A)*	*Compound gear D-E*	*Gear G*
1.	Arm fixed - gear *C* rotates through + 1 revolution	0	+ 1	$-\frac{T_C}{T_D}$	$-\frac{T_C}{T_D} \times \frac{T_E}{T_G}$
2.	Arm fixed - gear *C* rotates through + *x* revolutions	0	$+x$	$-x \times \frac{T_C}{T_D}$	$-x \times \frac{T_C}{T_D} \times \frac{T_E}{T_G}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \times \frac{T_C}{T_D}$	$y - x \times \frac{T_C}{T_D} \times \frac{T_E}{T_G}$

Since the gear *G* is fixed, therefore from the fourth row of the table,

$$y - x \times \frac{T_C}{T_D} \times \frac{T_E}{T_G} = 0 \quad \text{or} \quad y - x \times \frac{50}{20} \times \frac{35}{105} = 0$$

$\therefore$ $$y - \frac{5}{6}x = 0 \quad \text{...}(i)$$

Since the gear C is rigidly mounted on shaft A, therefore speed of gear C and shaft A is same. We know that speed of shaft A is 110 r.p.m., therefore from the fourth row of the table,

$$x + y = 100 \quad \text{...(ii)}$$

From equations (*i*) and (*ii*), $x = 60$, and $y = 50$

$\therefore$ Speed of shaft B = Speed of arm = $+ y$ = 50 r.p.m. anticlockwise **Ans.**

Example 13.11. *Fig. 13.15 shows diagrammatically a compound epicyclic gear train. Wheels A , D and E are free to rotate independently on spindle O, while B and C are compound and rotate together on spindle P, on the end of arm OP. All the teeth on different wheels have the same module. A has 12 teeth, B has 30 teeth and C has 14 teeth cut externally. Find the number of teeth on wheels D and E which are cut internally.*

If the wheel A is driven clockwise at 1 r.p.s. while D is driven counter clockwise at 5 r.p.s., determine the magnitude and direction of the angular velocities of arm OP and wheel E.

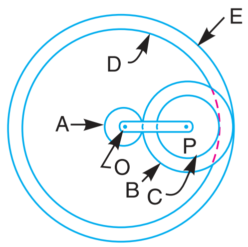


Fig. 13.15

Solution. Given : $T_A = 12$; $T_B = 30$; $T_C = 14$; $N_A = 1$ r.p.s. ; $N_D = 5$ r.p.s.

Number of teeth on wheels D and E

Let T_D and T_E be the number of teeth on wheels D and E respectively. Let d_A, d_B, d_C, d_D and d_E be the pitch circle diameters of wheels A, B, C, D and E respectively. From the geometry of the figure,

$$d_E = d_A + 2d_B \quad \text{and} \quad d_D = d_E - (d_B - d_C)$$

Since the number of teeth are proportional to their pitch circle diameters for the same module, therefore

$$T_E = T_A + 2T_B = 12 + 2 \times 30 = 72 \quad \textbf{Ans.}$$

and
$$T_D = T_E - (T_B - T_C) = 72 - (30 - 14) = 56 \quad \textbf{Ans.}$$

Magnitude and direction of angular velocities of arm OP and wheel E

The table of motions is drawn as follows :

Table 13.10. Table of motions.

Step No.	Conditions of motion	Revolutions of elements: Arm	Wheel A	Compound wheel B-C	Wheel D	Wheel E
1.	Arm fixed A rotated through – 1 revolution (*i.e.* 1 revolution clockwise)	0	– 1	$+\frac{T_A}{T_B}$	$+\frac{T_A}{T_B} \times \frac{T_C}{T_D}$	$+\frac{T_A}{T_B} \times \frac{T_B}{T_E} = +\frac{T_A}{T_E}$
2.	Arm fixed-wheel A rotated through $-x$ revolutions	0	$-x$	$+x \times \frac{T_A}{T_B}$	$+x \times \frac{T_A}{T_B} \times \frac{T_C}{T_D}$	$+x \times \frac{T_A}{T_E}$
3.	Add $-y$ revolutions to all elements	$-y$	$-y$	$-y$	$-y$	$-y$
4.	Total motion	$-y$	$-x-y$	$x \times \frac{T_A}{T_B} - y$	$x \times \frac{T_A}{T_B} \times \frac{T_C}{T_D} - y$	$x \times \frac{T_A}{T_E} - y$

Since the wheel A makes 1 r.p.s. clockwise, therefore from the fourth row of the table,

$$-x - y = -1 \quad \text{or} \quad x + y = 1 \qquad \text{...}(i)$$

Also, the wheel D makes 5 r.p.s. counter clockwise, therefore

$$x \times \frac{T_A}{T_B} \times \frac{T_C}{T_D} - y = 5 \quad \text{or} \quad x \times \frac{12}{30} \times \frac{14}{56} - y = 5$$

$$\therefore \quad 0.1\,x - y = 5 \qquad \text{...}(ii)$$

From equations **(*i*)** and **(*ii*),**

$$x = 5.45 \quad \text{and} \quad y = -4.45$$

∴ Angular velocity of arm OP

$$= -y = -(-4.45) = 4.45 \text{ r.p.s}$$

$$= 4.45 \times 2\,\pi = 27.964 \text{ rad/s (counter clockwise)} \quad \textbf{Ans.}$$

and angular velocity of wheel $E = x \times \frac{T_A}{T_E} - y = 5.45 \times \frac{12}{72} - (-4.45) = 5.36$ r.p.s.

$$= 5.36 \times 2\,\pi = 33.68 \text{ rad/s (counter clockwise)} \quad \textbf{Ans.}$$

Example 13.12. *An internal wheel B with 80 teeth is keyed to a shaft F. A fixed internal wheel C with 82 teeth is concentric with B. A compound wheel D-E gears with the two internal wheels; D has 28 teeth and gears with C while E gears with B. The compound wheels revolve freely on a pin which projects from a disc keyed to a shaft A co-axial with F. If the wheels have the same pitch and the shaft A makes 800 r.p.m., what is the speed of the shaft F ? Sketch the arrangement.*

Helicopter

Note : This picture is given as additional information and is not a direct example of the current chapter.

Solution. Given : $T_B = 80$; $T_C = 82$; $T_D = 28$; $N_A = 500$ r.p.m.

The arrangement is shown in Fig. 13.16.

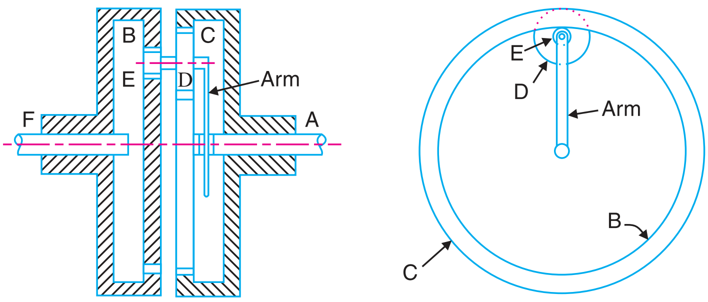


Fig. 13.16

First of all, let us find out the number of teeth on wheel E (T_E). Let d_B , d_C , d_D and d_E be the pitch circle diameter of wheels B, C, D and E respectively. From the geometry of the figure,

$$d_B = d_C - (d_D - d_E)$$

or $$d_E = d_B + d_D - d_C$$

Since the number of teeth are proportional to their pitch circle diameters for the same pitch, therefore

$$T_E = T_B + T_D - T_C = 80 + 28 - 82 = 26$$

The table of motions is given below :

Table 13.11. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Arm (or shaft A)	Wheel B (or shaft F)	Compound gear D-E	Wheel C
1.	Arm fixed - wheel *B* rotated through + 1 revolution (*i.e.* 1 revolution anticlockwise)	0	+ 1	$+\frac{T_B}{T_E}$	$+\frac{T_B}{T_E} \times \frac{T_D}{T_C}$
2.	Arm fixed - wheel *B* rotated through + *x* revolutions	0	$+x$	$+x \times \frac{T_B}{T_E}$	$+x \times \frac{T_B}{T_E} \times \frac{T_D}{T_C}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y + x \times \frac{T_B}{T_E}$	$y + x \times \frac{T_B}{T_E} \times \frac{T_D}{T_C}$

Since the wheel *C* is fixed, therefore from the fourth row of the table,

$$y + x \times \frac{T_B}{T_E} \times \frac{T_D}{T_C} = 0 \qquad \text{or} \qquad y + x \times \frac{80}{26} \times \frac{28}{82} = 0$$

$$\therefore \qquad y + 1.05\,x = 0 \qquad \text{...}(i)$$

Also, the shaft *A* (or the arm) makes 800 r.p.m., therefore from the fourth row of the table,

$$y = 800 \qquad \text{...}(ii)$$

From equations (*i*) and (*ii*),

$$x = -762$$

$\therefore$ Speed of shaft F = Speed of wheel $B = x + y = -762 + 800 = +38$ r.p.m.

$= 38$ r.p.m. (anticlockwise) **Ans.**

Example 13.13. *Fig. 13.17 shows an epicyclic gear train known as Ferguson's paradox. Gear A is fixed to the frame and is, therefore, stationary. The arm B and gears C and D are free to rotate on the shaft S. Gears A, C and D have 100, 101 and 99 teeth respectively. The planet gear has 20 teeth. The pitch circle diameters of all are the same so that the planet gear P meshes with all of them. Determine the revolutions of gears C and D for one revolution of the arm B.*

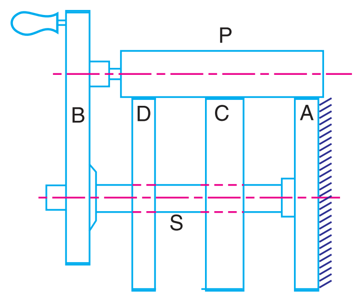


Fig. 13.17

Solution. Given : $T_A = 100$; $T_C = 101$; $T_D = 99$; $T_P = 20$

The table of motions is given below :

Table 13.12. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Arm B	*Gear A*	*Gear C*	*Gear D*
1.	Arm *B* fixed, gear *A* rotated through + 1 revolution (*i.e.* 1 revolution anticlockwise)	0	+ 1	$+\frac{T_A}{T_C}$	$+\frac{T_A}{T_C}\times\frac{T_C}{T_D}=+\frac{T_A}{T_D}$
2.	Arm *B* fixed, gear *A* rotated through + *x* revolutions	0	$+x$	$+x\times\frac{T_A}{T_C}$	$+x\times\frac{T_A}{T_D}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y+x\times\frac{T_A}{T_C}$	$y+x\times\frac{T_A}{T_D}$

The arm *B* makes one revolution, therefore

$$y = 1$$

Since the gear *A* is fixed, therefore from the fourth row of the table,

$$x + y = 0 \qquad \text{or} \qquad x = -y = -1$$

Let N_C and N_D = Revolutions of gears *C* and *D* respectively.

From the fourth row of the table, the revolutions of gear *C*,

$$N_C = y + x\times\frac{T_A}{T_C} = 1 - 1\times\frac{100}{101} = +\frac{1}{101} \quad \textbf{Ans.}$$

and the revolutions of gear *D*,

$$N_D = y + x\times\frac{T_A}{T_D} = 1 - \frac{100}{99} = -\frac{1}{99} \quad \textbf{Ans.}$$

From above we see that for one revolution of the arm *B*, the gear *C* rotates through 1/101 revolutions in the same direction and the gear *D* rotates through 1/99 revolutions in the opposite direction.

Example 13.14. *In the gear drive as shown in Fig. 13.18, the driving shaft A rotates at 300 r.p.m. in the clockwise direction, when seen from left hand. The shaft B is the driven shaft. The casing C is held stationary. The wheels E and H are keyed to the central vertical spindle and wheel F can rotate freely on this spindle. The wheels K and L are rigidly fixed to each other and rotate together freely on a pin fitted on the underside of F. The wheel L meshes with internal teeth on the casing C. The numbers of teeth on the different wheels are indicated within brackets in Fig. 13.18.*

Find the number of teeth on wheel C and the speed and direction of rotation of shaft B.

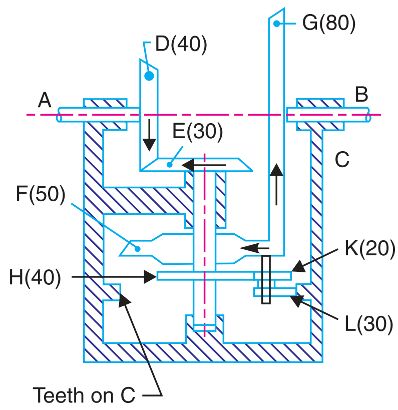


Fig. 13.18

Solution. Given : N_A = 300 r.p.m. (clockwise) ; $T_D = 40$; $T_B = 30$; $T_F = 50$; $T_G = 80$; $T_H = 40$; $T_K = 20$; $T_L = 30$

In the arrangement shown in Fig. 13.18, the wheels *D* and *G* are auxillary gears and do not form a part of the epicyclic gear train.

Speed of wheel E, $N_E = N_A \times \dfrac{T_D}{T_E} = 300 \times \dfrac{40}{30} = 400$ r.p.m. (clockwise)

Number of teeth on wheel C

Let T_C = Number of teeth on wheel C.

Assuming the same module for all teeth and since the pitch circle diameter is proportional to the number of teeth ; therefore from the geometry of Fig.13.18,

$$T_C = T_H + T_K + T_L = 40 + 20 + 30 = 90 \text{ Ans.}$$

Speed and direction of rotation of shaft B

The table of motions is given below. The wheel F acts as an arm.

Table 13.13. Table of motions.

Step No.	Conditions of motion	Revolutions of elements				
		Arm or wheel F	Wheel E	Wheel H	Compound wheel K-L	Wheel C
1.	Arm fixed-wheel E rotated through -1 revolution (*i.e.* 1 revolution clockwise)	0	-1	-1 ($\because$ E and H are on the same shaft)	$+\dfrac{T_H}{T_K}$	$+\dfrac{T_H}{T_K} \times \dfrac{T_L}{T_C}$
2.	Arm fixed-wheel E rotated through $-x$ revolutions	0	$-x$	$-x$	$+x \times \dfrac{T_H}{T_K}$	$+x \times \dfrac{T_H}{T_K} \times \dfrac{T_L}{T_C}$
3.	Add $-y$ revolutions to all elements	$-y$	$-y$	$-y$	$-y$	$-y$
4.	Total motion	$-y$	$-x-y$	$-x-y$	$x \times \dfrac{T_H}{T_K} - y$	$x \times \dfrac{T_H}{T_K} \times \dfrac{T_L}{T_C} - y$

Since the speed of wheel E is 400 r.p.m. (clockwise), therefore from the fourth row of the table,

$$-x - y = -400 \quad \text{or} \quad x + y = 400 \qquad \text{...(i)}$$

Also the wheel C is fixed, therefore

$$x \times \frac{T_H}{T_K} \times \frac{T_L}{T_C} - y = 0$$

or

$$x \times \frac{40}{20} \times \frac{30}{90} - y = 0$$

$$\therefore \quad \frac{2x}{3} - y = 0 \qquad \text{...(ii)}$$

From equations *(i)* and *(ii)*,

$$x = 240 \quad \text{and} \quad y = 160$$

$\therefore$ Speed of wheel F, $N_F = -y = -160$ r.p.m.

Since the wheel F is in mesh with wheel G, therefore speed of wheel G or speed of shaft B

$$= -N_F \times \frac{T_F}{T_G} = -\left(-160 \times \frac{50}{80}\right) = 100 \text{ r.p.m.}$$

...($\because$ Wheel G will rotate in opposite direction to that of wheel F.)

= 100 r.p.m. anticlockwise *i.e.* in opposite direction of shaft A. **Ans.**

Example 13.15. *Fig. 13.19 shows a compound epicyclic gear in which the casing C contains an epicyclic train and this casing is inside the larger casing D.*

Determine the velocity ratio of the output shaft B to the input shaft A when the casing D is held stationary. The number of teeth on various wheels are as follows :

Wheel on A = 80 ; Annular wheel on B = 160 ; Annular wheel on C = 100 ; Annular wheel on D = 120 ; Small pinion on F = 20 ; Large pinion on F = 66.

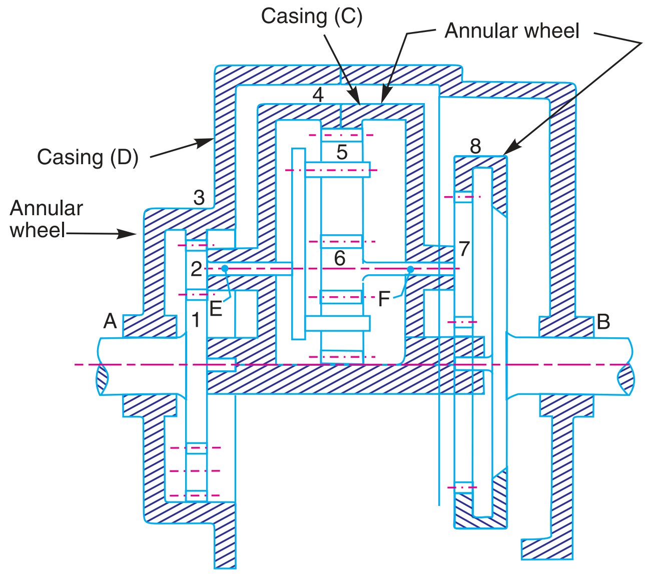


Fig. 13.19

Solution. Given : $T_1 = 80$; $T_8 = 160$; $T_4 = 100$; $T_3 = 120$; $T_6 = 20$; $T_7 = 66$

First of all, let us consider the train of wheel 1 (on *A*), wheel 2 (on *E*), annular wheel 3 (on *D*) and the arm *i.e.* casing *C*. Since the pitch circle diameters of wheels are proportional to the number of teeth, therefore from the geometry of Fig. 13.19,

$$T_1 + 2\,T_2 = T_3 \quad \text{or} \quad 80 + 2\,T_2 = 120$$

$$\therefore \qquad T_2 = 20$$

The table of motions for the train considered is given below :

Table 13.14. Table of motions.

Step No.	Conditons of motion	Revolutions of elements			
		Arm	Wheel 1	Wheel 2	Wheel 3
1.	Arm fixed - wheel 1 rotated through + 1 revolution (anticlockwise)	0	+ 1	$-\frac{T_1}{T_2}$	$-\frac{T_1}{T_2} \times \frac{T_2}{T_3} = -\frac{T_1}{T_3}$
2.	Arm fixed - wheel 1 rotated through + *x* revolutions	0	$+x$	$-x \times \frac{T_1}{T_2}$	$-x \times \frac{T_1}{T_3}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	y	$x+y$	$y - x \times \frac{T_1}{T_2}$	$y - x \times \frac{T_1}{T_3}$

Let us assume that wheel 1 makes 1 r.p.s. anticlockwise.

$\therefore \qquad x + y = 1 \qquad$ ...(*i*)

Also the wheel 3 is stationary, therefore from the fourth row of the table,

$$y - x \times \frac{T_1}{T_3} = 0 \quad \text{or} \quad y - x \times \frac{80}{120} = 0$$

$\therefore \qquad y - \frac{2}{3} x = 0 \qquad$ ...(*ii*)

From equations (*i*) and (*ii*), $x = 0.6$, and $y = 0.4$

$\therefore$ Speed of arm or casing $C = y = 0.4$ r.p.s.

and speed of wheel 2 or arm E $= y - x \times \frac{T_1}{T_2} = 0.4 - 0.6 \times \frac{80}{20} = -2$ r.p.s.

$= 2$ r.p.s. (clockwise)

Let us now consider the train of annular wheel 4 (on C), wheel 5 (on E), wheel 6 (on F) and arm E. We know that

$$T_6 + 2\,T_5 = T_4 \quad \text{or} \quad 20 + 2\,T_5 = 100$$

$\therefore \qquad T_5 = 40$

The table of motions is given below :

Table 13.15. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Arm E or wheel 2	Wheel 6	Wheel 5	Wheel 4
1.	Arm fixed, wheel 6 rotated through + 1 revolution	0	+ 1	$-\frac{T_6}{T_5}$	$-\frac{T_6}{T_5} \times \frac{T_5}{T_4} = -\frac{T_6}{T_4}$
2.	Arm fixed, wheel 6 rotated through + x_1 revolutions	0	x_1	$-x_1 \times \frac{T_6}{T_5}$	$-x_1 \times \frac{T_6}{T_4}$
3.	Add + y_1 revolutions to all elements	$+y_1$	$+y_1$	$+y_1$	$+y_1$
4.	Total motion	$+y_1$	$x_1 + y_1$	$y_1 - x_1 \times \frac{T_6}{T_5}$	$y_1 - x_1 \times \frac{T_6}{T_4}$

We know that speed of arm E = Speed of wheel 2 in the first train

$\therefore \qquad y_1 = -2 \qquad$ ...(*iii*)

Also speed of wheel 4 = Speed of arm or casing C in the first train

$\therefore \qquad y_1 - x_1 \times \frac{T_6}{T_4} = 0.4 \quad \text{or} \quad -2 - x_1 \times \frac{20}{100} = 0.4 \qquad$ ...(*iv*)

or $\qquad x_1 = (-2 - 0.4)\frac{100}{20} = -12$

∴ Speed of wheel 6 (or F)

$$= x_1 + y_1 = -12 - 2 = -14 \text{ r.p.s.} = 14 \text{ r.p.s. (clockwise)}$$

Now consider the train of wheels 6 and 7 (both on F), annular wheel 8 (on B) and the arm *i.e.* casing C. The table of motions is given below :

Table 13.16. Table of motions.

Step No.	Conditions of motion	Revolutions of elements		
		Arm	Wheel 8	Wheel 7
1.	Arm fixed, wheel 8 rotated through + 1 revolution	0	+ 1	$+\frac{T_8}{T_7}$
2.	Arm fixed, wheel 8 rotated through $+x_2$ revolutions	0	$+x_2$	$+x_2 \times \frac{T_8}{T_7}$
3.	Add $+y_2$ revolutions to all elements	$+y_2$	$+y_2$	$+y_2$
4.	Total motion	y_2	$x_2 + y_2$	$y_2 + x_2 \times \frac{T_8}{T_7}$

We know that the speed of C in the first train is 0.4 r.p.s., therefore

$$y_2 = 0.4 \qquad \text{...(v)}$$

Also the speed of wheel 7 is equal to the speed of F or wheel 6 in the second train, therefore

$$y_2 + x_2 \times \frac{T_8}{T_7} = -14 \quad \text{or} \quad 0.4 + x_2 \times \frac{160}{66} = -14 \qquad \text{...(vi)}$$

$$\therefore \quad x_2 = (-14 - 0.4)\frac{66}{160} = -5.94$$

∴ Speed of wheel 8 or of the shaft B

$$x_2 + y_2 = -5.94 + 0.4 = -5.54 \text{ r.p.s.} = 5.54 \text{ r.p.s. (clockwise)}$$

We have already assumed that the speed of wheel 1 or the shaft A is 1 r.p.s. anticlockwise

∴ Velocity ratio of the output shaft B to the input shaft A

$$= -5.54$$ **Ans.**

Note : The – ve sign shows that the two shafts A and B rotate in opposite directions.

13.10. Epicyclic Gear Train with Bevel Gears

The bevel gears are used to make a more compact epicyclic system and they permit a very high speed reduction with few gears. The useful application of the epicyclic gear train with bevel gears is found in Humpage's speed reduction gear and differential gear of an automobile as discussed below :

1. *Humpage's speed reduction gear.* The Humpage's speed reduction gear was originally designed as a substitute for back gearing of a lathe, but its use is now considerably extended to all kinds of workshop machines and also in electrical machinery. In Humpage's speed reduction gear, as shown in Fig. 13.20, the driving shaft X and the driven shaft Y are co-axial. The driving shaft carries a bevel gear A and driven shaft carries a bevel gear E. The bevel gear B meshes with gear A (also known as pinion) and a fixed gear C. The gear E meshes with gear D which is compound with gear B.

This compound gear *B*-*D* is mounted on the arm or spindle *F* which is rigidly connected with a hollow sleeve *G*. The sleeve revolves freely loose on the axes of the driving and driven shafts.

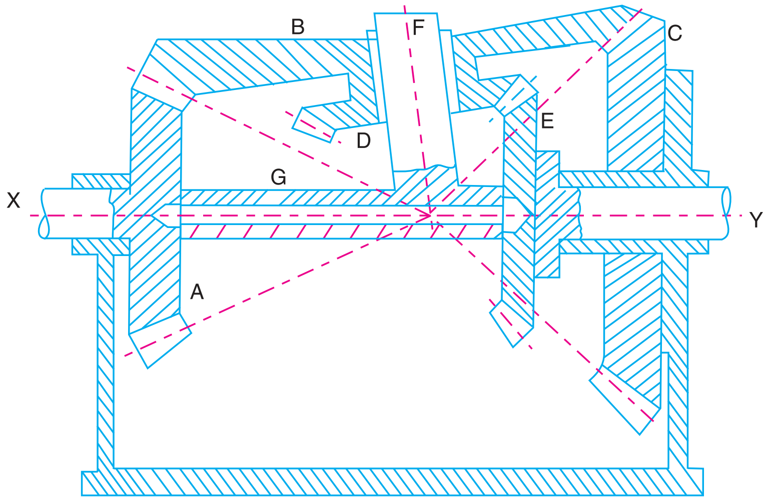


Fig. 13.20. Humpage's speed reduction gear.

2. *Differential gear of an automobile*. The differential gear used in the rear drive of an automobile is shown in Fig. 13.21. Its function is

(*a*) to transmit motion from the engine shaft to the rear driving wheels, and

(*b*) to rotate the rear wheels at different speeds while the automobile is taking a turn.

As long as the automobile is running on a straight path, the rear wheels are driven directly by the engine and speed of both the wheels is same. But when the automobile is taking a turn, the outer wheel will run faster than the * inner wheel because at that time the outer rear wheel has to cover more distance than the inner rear wheel. This is achieved by epicyclic gear train with bevel gears as shown in Fig. 13.21.

The bevel gear *A* (known as pinion) is keyed to the propeller shaft driven from the engine shaft through universal coupling. This gear *A* drives the gear *B* (known as crown gear) which rotates freely on the axle *P*. Two equal gears *C* and *D* are mounted on two separate parts *P* and *Q* of the rear axles respectively. These gears, in turn, mesh with equal pinions *E* and *F* which can rotate freely on the spindle provided on the arm attached to gear *B*.

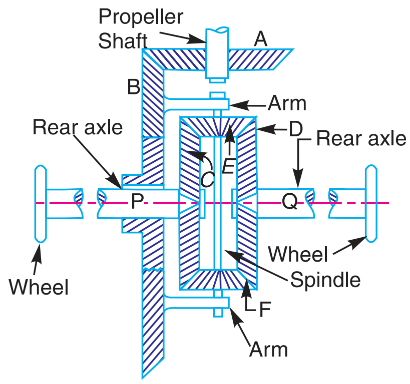


Fig. 13.21. Differential gear of an automobile.

When the automobile runs on a straight path, the gears *C* and *D* must rotate together. These gears are rotated through the spindle on the gear *B*. The gears *E* and *F* do not rotate on the spindle. But when the automobile is taking a turn, the inner rear wheel should have lesser speed than the outer rear wheel and due to relative speed of the inner and outer gears *D* and *C*, the gears *E* and *F* start rotating about the spindle axis and at the same time revolve about the axle axis.

Due to this epicyclic effect, the speed of the inner rear wheel decreases by a certain amount and the speed of the outer rear wheel increases, by the same amount. This may be well understood by drawing the table of motions as follows :

* This difficulty does not arise with the front wheels as they are greatly used for steering purposes and are mounted on separate axles and can run freely at different speeds.

Table 13.17. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Gear B	*Gear C*	*Gear E*	*Gear D*
1.	Gear *B* fixed-Gear *C* rotated through + 1 revolution (*i.e.* 1 revolution anticlockwise)	0	+ 1	$+\frac{T_C}{T_E}$	$-\frac{T_C}{T_E}\times\frac{T_E}{T_D}=-1$ $(\because T_C=T_D)$
2.	Gear *B* fixed-Gear *C* rotated through + *x* revolutions	0	$+x$	$+x\times\frac{T_C}{T_E}$	$-x$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y+x\times\frac{T_C}{T_E}$	$y-x$

From the table, we see that when the gear *B*, which derives motion from the engine shaft, rotates at *y* revolutions, then the speed of inner gear *D* (or the rear axle *Q*) is less than *y* by *x* revolutions and the speed of the outer gear *C* (or the rear axle *P*) is greater than *y* by *x* revolutions. In other words, the two parts of the rear axle and thus the two wheels rotate at two different speeds. We also see from the table that the speed of gear *B* is the mean of speeds of the gears *C* and *D*.

Example 13.16. *Two bevel gears A and B (having 40 teeth and 30 teeth) are rigidly mounted on two co-axial shafts X and Y. A bevel gear C (having 50 teeth) meshes with A and B and rotates freely on one end of an arm. At the other end of the arm is welded a sleeve and the sleeve is riding freely loose on the axes of the shafts X and Y. Sketch the arrangement.*

If the shaft X rotates at 100 r.p.m. clockwise and arm rotates at 100 r.p.m.anitclockwise, find the speed of shaft Y.

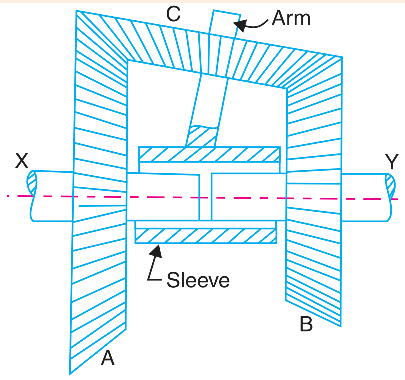


Fig. 13.22

Solution. Given : $T_A = 40$; $T_B = 30$; $T_C = 50$; $N_X = N_A = 100$ r.p.m. (clockwise) ; Speed of arm = 100 r.p.m. (anticlockwise)

The arangement is shown in Fig. 13.22.

The table of motions is drawn as below :

Table 13.18. Table of motions.

Step No.	*Conditions of motion*	*Revolutions of elements*			
		Arm	*Gear A*	*Gear C*	*Gear B*
1.	Arm *B* fixed, gear *A* rotated through + 1 revolution (*i.e.* 1 revolution anticlockwise)	0	+ 1	$\pm^{*}\frac{T_A}{T_C}$	$-\frac{T_A}{T_C}\times\frac{T_C}{T_B}=-\frac{T_A}{T_B}$
2.	Arm *B* fixed, gear *A* rotated through + *x* revolutions	0	$+x$	$\pm x\times\frac{T_A}{T_C}$	$-x\times\frac{T_A}{T_B}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y\pm x\times\frac{T_A}{T_C}$	$y-x\times\frac{T_A}{T_B}$

* The ± sign is given to the motion of the wheel *C* because it is in a different plane. So we cannot indicate the direction of its motion specifically, *i.e.* either clockwise or anticlockwise.

Since the speed of the arm is 100 r.p.m. anticlockwise, therefore from the fourth row of the table,

$$y = +100$$

Also, the speed of the driving shaft X or gear A is 100 r.p.m. clockwise.

$\therefore \quad x + y = -100 \quad$ or $\quad x = -y - 100 = -100 - 100 = -200$

$\therefore$ Speed of the driven shaft *i.e.* shaft Y,

$$N_Y = \text{Speed of gear } B = y - x \times \frac{T_A}{T_B} = 100 - \left(-200 \times \frac{40}{30}\right)$$

$$= +366.7 \text{ r.p.m.} = 366.7 \text{ r.p.m. (anticlockwise)} \textbf{ Ans.}$$

Example 13.17. *In a gear train, as shown in Fig. 13.23, gear B is connected to the input shaft and gear F is connected to the output shaft. The arm A carrying the compound wheels D and E, turns freely on the output shaft. If the input speed is 1000 r.p.m. counter- clockwise when seen from the right, determine the speed of the output shaft under the following conditions :*

1. When gear C is fixed, and 2. when gear C is rotated at 10 r.p.m. counter clockwise.

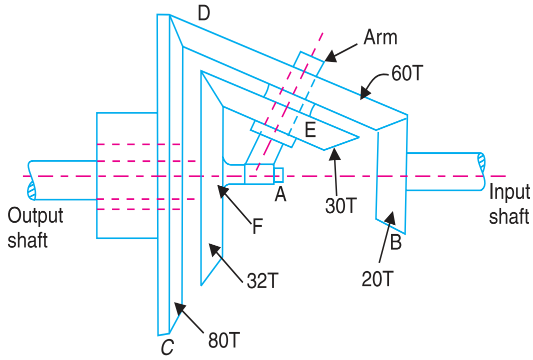


Fig. 13.23

Solution. Given : $T_B = 20$; $T_C = 80$; $T_D = 60$; $T_E = 30$; $T_F = 32$; $N_B = 1000$ r.p.m. (counter-clockwise)

The table of motions is given below :

Table 13.19. Table of motions.

Step No.	Conditions of motion	Revolutions of elements				
		Arm A	Gear B (or input shaft)	Compound wheel D-E	Gear C	Gear F (or output shaft)
1.	Arm fixed, gear B rotated through + 1 revolution (*i.e.* 1 revolution anticlockwise)	0	+ 1	$+\frac{T_B}{T_D}$	$-\frac{T_B}{T_D} \times \frac{T_D}{T_C} = -\frac{T_B}{T_C}$	$-\frac{T_B}{T_D} \times \frac{T_E}{T_F}$
2.	Arm fixed, gear B rotated through $+x$ revolutions	0	$+x$	$+x \times \frac{T_B}{T_D}$	$-x \times \frac{T_B}{T_C}$	$-x \times \frac{T_B}{T_D} \times \frac{T_E}{T_F}$
3.	Add $+y$ revolutions to all elements	$+y$	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y + x \times \frac{T_B}{T_D}$	$y - x \times \frac{T_B}{T_C}$	$y - x \times \frac{T_B}{T_D} \times \frac{T_E}{T_F}$

1. Speed of the output shaft when gear C is fixed

Since the gear C is fixed, therefore from the fourth row of the table,

$$y - x \times \frac{T_B}{T_C} = 0 \quad \text{or} \quad y - x \times \frac{20}{80} = 0$$

$\therefore \quad y - 0.25\,x = 0 \qquad ...(i)$

We know that the input speed (or the speed of gear B) is 1000 r.p.m. counter clockwise, therefore from the fourth row of the table,

$x + y = +\,1000 \qquad ...(ii)$

From equations (*i*) and (*ii*), $x = +\,800$, and $y = +\,200$

$$\therefore \text{ Speed of output shaft } = \text{Speed of gear } F = y - x \times \frac{T_B}{T_D} \times \frac{T_E}{T_F}$$

$$= 200 - 800 \times \frac{20}{80} \times \frac{30}{32} = 200 - 187.5 = 12.5 \text{ r.p.m.}$$

= 12.5 r.p.m. (counter clockwise) **Ans.**

2. Speed of the output shaft when gear C is rotated at 10 r.p.m. counter clockwise

Since the gear C is rotated at 10 r.p.m. counter clockwise, therefore from the fourth row of the table,

$$y - x \times \frac{T_B}{T_C} = +10 \quad \text{or} \quad y - x \times \frac{20}{80} = 10$$

$\therefore \quad y - 0.25\,x = 10 \qquad ...(iii)$

From equations (*ii*) and (*iii*),

$x = 792$, and $y = 208$

$\therefore$ Speed of output shaft

$$= \text{Speed of gear } F = y - x \times \frac{T_B}{T_D} \times \frac{T_E}{T_F} = 208 - 792 \times \frac{20}{80} \times \frac{30}{32}$$

= 208 – 185.6 = 22.4 r.p.m. = 22.4 r.p.m. (counter clockwise) **Ans.**

Example 13.18. *Fig. 13.24 shows a differential gear used in a motor car. The pinion A on the propeller shaft has 12 teeth and gears with the crown gear B which has 60 teeth. The shafts P and Q form the rear axles to which the road wheels are attached. If the propeller shaft rotates at 1000 r.p.m. and the road wheel attached to axle Q has a speed of 210 r.p.m. while taking a turn, find the speed of road wheel attached to axle P.*

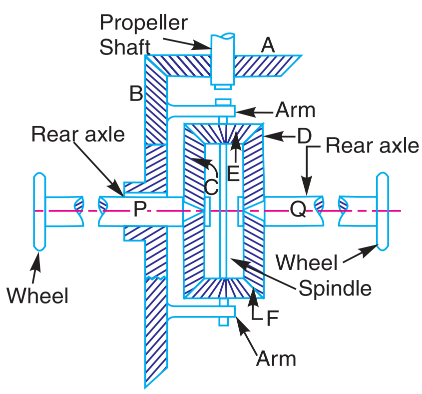


Fig. 13.24

Solution. Given : $T_A = 12$; $T_B = 60$; $N_A = 1000$ r.p.m. ; $N_Q = N_D = 210$ r.p.m.

Since the propeller shaft or the pinion A rotates at 1000 r.p.m., therefore speed of crown gear B,

$$N_B = N_A \times \frac{T_A}{T_B} = 1000 \times \frac{12}{60}$$

$$= 200 \text{ r.p.m.}$$

The table of motions is given below :

Table 13.20. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Gear B	Gear C	Gear E	Gear D
1.	Gear B fixed-Gear C rotated through + 1 revolution (*i.e.* 1 revolution anticlockwise)	0	+ 1	$+\frac{T_C}{T_E}$	$-\frac{T_C}{T_E}\times\frac{T_E}{T_D}=-1$ $(\because T_C = T_D)$
2.	Gear B fixed-Gear C rotated through $+x$ revolutions	0	$+x$	$+x\times\frac{T_C}{T_E}$	$-x$
3.	Add $+y$ revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y+x\times\frac{T_C}{T_E}$	$y-x$

Since the speed of gear B is 200 r.p.m., therefore from the fourth row of the table,

$$y = 200 \qquad \text{...(i)}$$

Also, the speed of road wheel attached to axle Q or the speed of gear D is 210 r.p.m., therefore from the fourth row of the table,

$$y - x = 210 \quad \text{or} \quad x = y - 210 = 200 - 210 = -10$$

∴ Speed of road wheel attached to axle P

$$= \text{Speed of gear } C = x + y$$

$$= -10 + 200 = 190 \text{ r.p.m.} \quad \textbf{Ans.}$$

13.11. Torques in Epicyclic Gear Trains

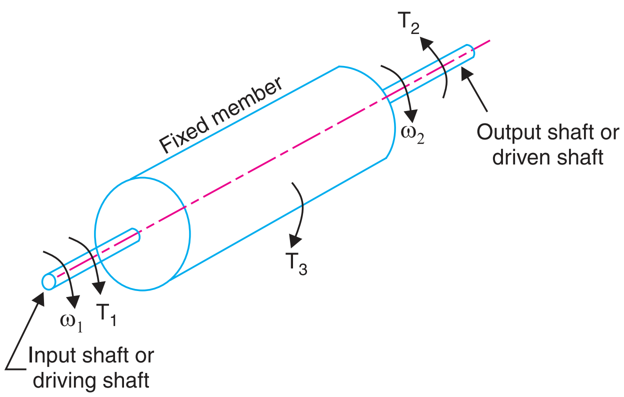


Fig. 13.25. Torques in epicyclic gear trains.

When the rotating parts of an epicyclic gear train, as shown in Fig. 13.25, have no angular acceleration, the gear train is kept in equilibrium by the three externally applied torques, *viz.*

1. Input torque on the driving member (T_1),

2. Output torque or resisting or load torque on the driven member (T_2),

3. Holding or braking or fixing torque on the fixed member (T_3).

The net torque applied to the gear train must be zero. In other words,

$$T_1 + T_2 + T_3 = 0 \qquad \text{...}(i)$$

$$\therefore \quad F_1.r_1 + F_2.r_2 + F_3.r_3 = 0 \qquad \text{...}(ii)$$

where F_1, F_2 and F_3 are the corresponding externally applied forces at radii r_1, r_2 and r_3.

Further, if ω_1, ω_2 and ω_3 are the angular speeds of the driving, driven and fixed members respectively, and the friction be neglected, then the net kinetic energy dissipated by the gear train must be zero, *i.e.*

$$T_1.\omega_1 + T_2.\omega_2 + T_3.\omega_3 = 0 \qquad \text{...}(iii)$$

But, for a fixed member, $\omega_3 = 0$

$$\therefore \quad T_1.\omega_1 + T_2.\omega_2 = 0 \qquad \text{...}(iv)$$

Notes : 1. From equations *(i)* and *(iv)*, the holding or braking torque T_3 may be obtained as follows :

$$T_2 = -T_1 \times \frac{\omega_1}{\omega_2} \qquad \text{...[From equation } (iv)]$$

and

$$T_3 = -(T_1 + T_2) \qquad \text{...[From equation } (i)]$$

$$= T_1\left(\frac{\omega_1}{\omega_2} - 1\right) = T_1\left(\frac{N_1}{N_2} - 1\right)$$

2. When input shaft (or driving shaft) and output shaft (or driven shaft) rotate in the same direction, then the input and output torques will be in opposite directions. Similarly, when the input and output shafts rotate in opposite directions, then the input and output torques will be in the same direction.

Example 13.19. *Fig. 13.26 shows an epicyclic gear train. Pinion A has 15 teeth and is rigidly fixed to the motor shaft. The wheel B has 20 teeth and gears with A and also with the annular fixed wheel E. Pinion C has 15 teeth and is integral with B (B, C being a compound gear wheel). Gear C meshes with annular wheel D, which is keyed to the machine shaft. The arm rotates about the same shaft on which A is fixed and carries the compound wheel B, C. If the motor runs at 1000 r.p.m., find the speed of the machine shaft. Find the torque exerted on the machine shaft, if the motor develops a torque of 100 N-m.*

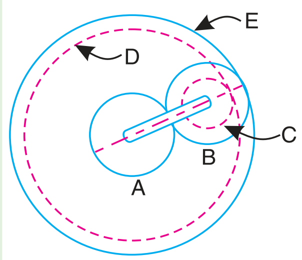


Fig. 13.26

Solution. Given : $T_A = 15$; $T_B = 20$; $T_C = 15$; $N_A = 1000$ r.p.m.; Torque developed by motor (or pinion *A*) = 100 N-m

First of all, let us find the number of teeth on wheels *D* and *E*. Let T_D and T_E be the number of teeth on wheels *D* and *E* respectively. Let d_A, d_B, d_C, d_D and d_E be the pitch circle diameters of wheels *A*, *B*, *C*, *D* and *E* respectively. From the geometry of the figure,

$$d_E = d_A + 2\,d_B \quad \text{and} \quad d_D = d_E - (d_B - d_C)$$

Since the number of teeth are proportional to their pitch circle diameters, therefore,

$$T_E = T_A + 2\,T_B = 15 + 2 \times 20 = 55$$

and

$$T_D = T_E - (T_B - T_C) = 55 - (20 - 15) = 50$$

Speed of the machine shaft

The table of motions is given below :

Table 13.21. Table of motions.

Step No.	Conditions of motion	Revolutions of elements				
		Arm	Pinion A	Compound wheel B-C	Wheel D	Wheel E
1.	Arm fixed-pinion A rotated through + 1 revolution (anticlockwise)	0	+ 1	$-\frac{T_A}{T_B}$	$-\frac{T_A}{T_B}\times\frac{T_C}{T_D}$	$-\frac{T_A}{T_B}\times\frac{T_B}{T_E}=-\frac{T_A}{T_E}$
2.	Arm fixed-pinion A rotated through + x revolutions	0	$+x$	$-x\times\frac{T_A}{T_B}$	$-x\times\frac{T_A}{T_B}\times\frac{T_C}{T_D}$	$-x\times\frac{T_A}{T_E}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y-x\times\frac{T_A}{T_B}$	$y-x\times\frac{T_A}{T_B}\times\frac{T_C}{T_D}$	$y-x\times\frac{T_A}{T_E}$

We know that the speed of the motor or the speed of the pinion A is 1000 r.p.m. Therefore

$$x + y = 1000 \qquad \text{...(i)}$$

Also, the annular wheel E is fixed, therefore

$$y - x\times\frac{T_A}{T_E} = 0 \quad \text{or} \quad y = x\times\frac{T_A}{T_E} = x\times\frac{15}{55} = 0.273\,x \qquad \text{...(ii)}$$

From equations (*i*) and (*ii*),

$$x = 786 \quad \text{and} \quad y = 214$$

∴ Speed of machine shaft = Speed of wheel D,

$$N_D = y - x\times\frac{T_A}{T_B}\times\frac{T_C}{T_D} = 214 - 786\times\frac{15}{20}\times\frac{15}{50} = +37.15 \text{ r.p.m.}$$

$$= 37.15 \text{ r.p.m. (anticlockwise)}$$ **Ans.**

Torque exerted on the machine shaft

We know that

Torque developed by motor × Angular speed of motor

= Torque exerted on machine shaft × Angular speed of machine shaft

or $100\times\omega_A$ = Torque exerted on machine shaft × ω_D

∴ Torque exerted on machine shaft

$$= 100\times\frac{\omega_A}{\omega_D} = 100\times\frac{N_A}{N_D} = 100\times\frac{1000}{37.15} = 2692 \text{ N-m}$$ **Ans.**

Example 13.20. *An epicyclic gear train consists of a sun wheel S, a stationary internal gear E and three identical planet wheels P carried on a star- shaped planet carrier C. The size of different toothed wheels are such that the planet carrier C rotates at 1/5th of the speed of the sunwheel S. The minimum number of teeth on any wheel is 16. The driving torque on the sun wheel is 100 N-m. Determine :* ***1.*** *number of teeth on different wheels of the train, and* ***2.*** *torque necessary to keep the internal gear stationary.*

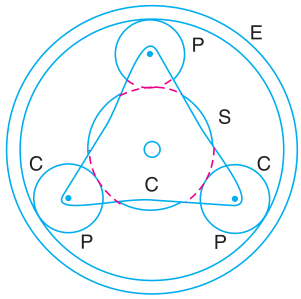


Fig. 13.27

Solution. Given : $N_C = \dfrac{N_S}{5}$

1. *Number of teeth on different wheels*

The arrangement of the epicyclic gear train is shown in Fig. 13.27. Let T_S and T_E be the number of teeth on the sun wheel *S* and the internal gear *E* respectively. The table of motions is given below :

Table 13.22. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Planet carrier C	*Sun wheel S*	*Planet wheel P*	*Internal gear E*
1.	Planet carrier *C* fixed, sunwheel *S* rotates through + 1 revolution (*i.e.* 1 rev. anticlockwise)	0	+ 1	$-\dfrac{T_S}{T_P}$	$-\dfrac{T_S}{T_P} \times \dfrac{T_P}{T_E} = -\dfrac{T_S}{T_E}$
2.	Planet carrier *C* fixed, sunwheel *S* rotates through + *x* revolutions	0	$+x$	$-x \times \dfrac{T_S}{T_P}$	$-x \times \dfrac{T_S}{T_E}$
3.	Add + *y* revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \times \dfrac{T_S}{T_P}$	$y - x \times \dfrac{T_S}{T_E}$

We know that when the sunwheel *S* makes 5 revolutions, the planet carrier *C* makes 1 revolution. Therefore from the fourth row of the table,

$$y = 1, \quad \text{and} \quad x + y = 5 \quad \text{or} \quad x = 5 - y = 5 - 1 = 4$$

Since the gear *E* is stationary, therefore from the fourth row of the table,

$$y - x \times \frac{T_S}{T_E} = 0 \quad \text{or} \quad 1 - 4 \times \frac{T_S}{T_E} = 0 \quad \text{or} \quad \frac{T_S}{T_E} = \frac{1}{4}$$

$\therefore$ $T_E = 4T_S$

Since the minimum number of teeth on any wheel is 16, therefore let us take the number of teeth on sunwheel, $T_S = 16$

$\therefore$ $T_E = 4\,T_S = 64$ **Ans.**

Let d_S, d_P and d_E be the pitch circle diameters of wheels *S*, *P* and *E* respectively. Now from the geometry of Fig. 13.27,

$$d_S + 2\,d_P = d_E$$

Assuming the module of all the gears to be same, the number of teeth are proportional to their pitch circle diameters.

$$T_S + 2\,T_P = T_E \quad \text{or} \quad 16 + 2\,T_P = 64 \quad \text{or} \quad T_P = 24 \textbf{ Ans.}$$

2. Torque necessary to keep the internal gear stationary

We know that

Torque on S × Angular speed of S

= Torque on C × Angular speed of C

$$100 \times \omega_S = \text{Torque on } C \times \omega_C$$

$$\therefore \quad \text{Torque on } C = 100 \times \frac{\omega_S}{\omega_C} = 100 \times \frac{N_S}{N_C} = 100 \times 5 = 500 \text{ N-m}$$

∴ Torque necessary to keep the internal gear stationary

= 500 – 100 = 400 N-m **Ans.**

Example 13.21. *In the epicyclic gear train, as shown in Fig. 13.28, the driving gear A rotating in clockwise direction has 14 teeth and the fixed annular gear C has 100 teeth. The ratio of teeth in gears E and D is 98 : 41. If 1.85 kW is supplied to the gear A rotating at 1200 r.p.m., find :* ***1.*** *the speed and direction of rotation of gear E, and* ***2.*** *the fixing torque required at C, assuming 100 per cent efficiency throughout and that all teeth have the same pitch.*

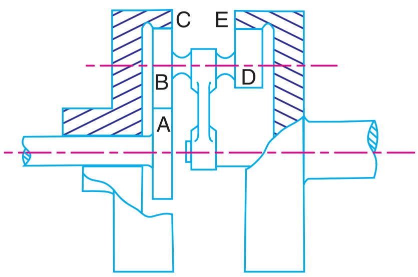


Fig. 13.28

Solution. Given : $T_A = 14$; $T_C = 100$; $T_E / T_D = 98 / 41$; $P_A = 1.85$ kW = 1850 W ; $N_A = 1200$ r.p.m.

Let d_A, d_B and d_C be the pitch circle diameters of gears A, B and C respectively. From Fig. 13.28,

$$d_A + 2\,d_B = d_C$$

Gears are extensively used in trains for power transmission.

Since teeth of all gears have the same pitch and the number of teeth are proportional to their pitch circle diameters, therefore

$$T_A + 2T_B = T_C \qquad \text{or} \qquad T_B = \frac{T_C - T_A}{2} = \frac{100 - 14}{2} = 43$$

The table of motions is now drawn as below :

Table 13.23. Table of motions.

Step No.	Conditions of motion	Revolutions of elements				
		Arm	Gear A	Compound gear B-D	Gear C	Gear E
1.	Arm fixed-Gear *A* rotated through – 1 revolution (*i.e.* 1 revolution clockwise)	0	-1	$+\frac{T_A}{T_B}$	$+\frac{T_A}{T_B}\times\frac{T_B}{T_C} = +\frac{T_A}{T_C}$	$+\frac{T_A}{T_B}\times\frac{T_D}{T_E}$
2.	Arm fixed-Gear *A* rotated through – *x* revolutions	0	$-x$	$+x\times\frac{T_A}{T_B}$	$+x\times\frac{T_A}{T_C}$	$+x\times\frac{T_A}{T_B}\times\frac{T_D}{T_E}$
3.	Add – *y* revolutions to all elements	$-y$	$-y$	$-y$	$-y$	$-y$
4.	Total motion	$-y$	$-y-x$	$-y+x\times\frac{T_A}{T_B}$	$-y+x\times\frac{T_A}{T_C}$	$-y+x\times\frac{T_A}{T_B}\times\frac{T_D}{T_E}$

Since the annular gear *C* is fixed, therefore from the fourth row of the table,

$$-y + x\times\frac{T_A}{T_C} = 0 \qquad \text{or} \qquad -y + x\times\frac{14}{100} = 0$$

$$\therefore \qquad -y + 0.14\,x = 0 \qquad \text{...(i)}$$

Also, the gear *A* is rotating at 1200 r.p.m., therefore

$$-x - y = 1200 \qquad \text{...(ii)}$$

From equations (*i*) and (*ii*), $x = -1052.6$, and $y = -147.4$

1. *Speed and direction of rotation of gear E*

From the fourth row of the table, speed of gear *E*,

$$N_E = -y + x\times\frac{T_A}{T_B}\times\frac{T_D}{T_E} = 147.4 - 1052.6\times\frac{14}{43}\times\frac{41}{98}$$

$$= 147.4 - 143.4 = 4 \text{ r.p.m.}$$

$$= 4 \text{ r.p.m. (anticlockwise)}$$ **Ans.**

2. *Fixing torque required at C*

We know that torque on $A = \frac{P_A \times 60}{2\pi N_A} = \frac{1850\times 60}{2\pi\times 1200} = 14.7$ N-m

Since the efficiency is 100 per cent throughout, therefore the power available at *E* (P_E) will be equal to power supplied at *A* (P_A).

$\therefore$ Torque on E $= \dfrac{P_A \times 60}{2\pi \times N_E} = \dfrac{1850 \times 60}{2\pi \times 4} = 4416$ N-m

$\therefore$ Fixing torque required at C

$= 4416 - 14.7 = 4401.3$ N-m **Ans.**

Example 13.22. *An over drive for a vehicle consists of an epicyclic gear train, as shown in Fig. 13.29, with compound planets B-C. B has 15 teeth and meshes with an annulus A which has 60 teeth. C has 20 teeth and meshes with the sunwheel D which is fixed. The annulus is keyed to the propeller shaft Y which rotates at 740 rad /s. The spider which carries the pins upon which the planets revolve, is driven directly from main gear box by shaft X, this shaft being relatively free to rotate with respect to wheel D. Find the speed of shaft X, when all the teeth have the same module.*

When the engine develops 130 kW, what is the holding torque on the wheel D ? Assume 100 per cent efficiency throughout.

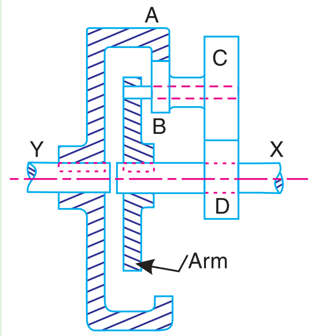


Fig. 13.29

Solution. Given : $T_B = 15$; $T_A = 60$; $T_C = 20$; $\omega_Y = \omega_A = 740$ rad /s ; $P = 130$ kW $= 130 \times 10^3$ W

First of all, let us find the number of teeth on the sunwheel D (T_D). Let d_A, d_B, d_C and d_D be the pitch circle diameters of wheels A, B, C and D respectively. From Fig. 13.29,

$$\frac{d_D}{2} + \frac{d_C}{2} + \frac{d_B}{2} = \frac{d_A}{2} \quad \text{or} \quad d_D + d_C + d_B = d_A$$

Since the module is same for all teeth and the number of teeth are proportional to their pitch circle diameters, therefore

$$T_D + T_C + T_B = T_A \quad \text{or} \quad T_D = T_A - (T_C + T_B) = 60 - (20 + 15) = 25$$

The table of motions is given below :

Table 13.24. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Arm (or shaft X)	Wheel D	Compound wheel C-B	Wheel A (or shaft Y)
1.	Arm fixed-wheel D rotated through + 1 revolution (anticlockwise)	0	+ 1	$-\dfrac{T_D}{T_C}$	$-\dfrac{T_D}{T_C} \times \dfrac{T_B}{T_A}$
2.	Arm fixed-wheel D rotated through + x revolutions	0	$+x$	$-x \times \dfrac{T_D}{T_C}$	$-x \times \dfrac{T_D}{T_C} \times \dfrac{T_B}{T_A}$
3.	Add + y revolutions to all elements	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \times \dfrac{T_D}{T_C}$	$y - x \times \dfrac{T_D}{T_C} \times \dfrac{T_B}{T_A}$

Since the shaft Y or wheel A rotates at 740 rad/s, therefore

$$y - x \times \frac{T_D}{T_C} \times \frac{T_B}{T_A} = 740 \quad \text{or} \quad y - x \times \frac{25}{20} \times \frac{15}{60} = 740$$

$$y - 0.3125\, x = 740 \qquad \text{...}(i)$$

Also the wheel D is fixed, therefore

$$x + y = 0 \quad \text{or} \quad y = -x \qquad \text{...}(ii)$$

From equations (*i*) and (*ii*),

$$x = -563.8 \quad \text{and} \quad y = 563.8$$

Speed of shaft X

Since the shaft X will make the same number of revolutions as the arm, therefore

Speed of shaft X, ω_X = Speed of arm = y = 563.8 rad/s **Ans.**

Holding torque on wheel D

We know that torque on $A = P/\omega_A = 130 \times 10^3 / 740 = 175.7$ N-m

and Torque on $X = P/\omega_X = 130 \times 10^3/563.8 = 230.6$ N-m

$\therefore$ Holding torque on wheel D

$$= 230.6 - 175.7 = 54.9 \text{ N-m} \textbf{ Ans.}$$

Example 13.23. *Fig. 13.30 shows some details of a compound epicyclic gear drive where I is the driving or input shaft and O is the driven or output shaft which carries two arms A and B rigidly fixed to it. The arms carry planet wheels which mesh with annular wheels P and Q and the sunwheels X and Y. The sun wheel X is a part of Q. Wheels Y and Z are fixed to the shaft I. Z engages with a planet wheel carried on Q and this planet wheel engages the fixed annular wheel R. The numbers of teeth on the wheels are :*

P = 114, Q = 120, R = 120, X = 36, Y = 24 and Z = 30.

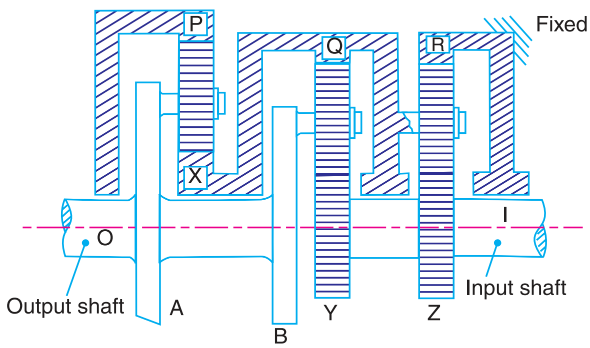


Fig. 13.30.

The driving shaft I makes 1500 r.p.m.clockwise looking from our right and the input at I is 7.5 kW.

1. *Find the speed and direction of rotation of the driven shaft O and the wheel P.*

2. *If the mechanical efficiency of the drive is 80%, find the torque tending to rotate the fixed wheel R.*

Solution. Given : $T_P = 144$; $T_Q = 120$; $T_R = 120$; $T_X = 36$; $T_Y = 24$; $T_Z = 30$; $N_I = 1500$ r.p.m. (clockwise) ; $P = 7.5$ kW = 7500 W ; $\eta = 80\% = 0.8$

First of all, consider the train of wheels Z, R and Q (arm). The revolutions of various wheels are shown in the following table.

Table 13.25. Table of motions.

Step No.	Conditions of motion	Revolutions of elements		
		Q (Arm)	Z (also I)	R (Fixed)
1.	Arm fixed-wheel Z rotates through + 1 revolution (anticlockwise)	0	+ 1	$-\frac{T_Z}{T_R}$
2.	Arm fixed-wheel Z rotates through $+x$ revolutions	0	$+x$	$-x \times \frac{T_Z}{T_R}$
3.	Add $+y$ revolutions to all elements	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y - x \times \frac{T_Z}{T_R}$

Since the driving shaft I as well as wheel Z rotates at 1500 r.p.m. clockwise, therefore

$$x + y = -1500 \quad \text{...(i)}$$

Also, the wheel R is fixed. Therefore

$$y - x \times \frac{T_Z}{T_R} = 0 \quad \text{or} \quad y = x \times \frac{T_Z}{T_R} = x \times \frac{30}{120} = 0.25\,x \quad \text{...(ii)}$$

From equations *(i)* and *(ii)*,

$$x = -1200, \quad \text{and} \quad y = -300$$

Now consider the train of wheels Y, Q, arm A, wheels P and X. The revolutions of various elements are shown in the following table.

Table 13.26. Table of motions.

Step No.	Conditions of motion	Revolutions of elements			
		Arm A, B and Shaft O	Wheel Y	Compound wheel Q-X	Wheel P
1.	Arm A fixed-wheel Y rotates through + 1 revolution (anticlockwise)	0	+ 1	$-\frac{T_Y}{T_Q}$	$+\frac{T_Y}{T_Q} \times \frac{T_X}{T_P}$
2.	Arm A fixed-wheel Y rotates through $+x_1$ revolutions	0	$+x_1$	$-x_1 \times \frac{T_Y}{T_Q}$	$+x_1 \times \frac{T_Y}{T_Q} \times \frac{T_X}{T_P}$
3.	Add $+y_1$ revolutions to all elements	$+y_1$	$+y_1$	$+y_1$	$+y_1$
4.	Total motion	$+y_1$	x_1+y_1	$y_1 - x_1 \times \frac{T_Y}{T_Q}$	$y_1 + x_1 \times \frac{T_Y}{T_Q} \times \frac{T_X}{T_P}$

Since the speed of compound wheel Q-X is same as that of Q, therefore

$$y_1 - x_1 \times \frac{T_Y}{T_Q} = y = -300$$

or

$$y_1 - x_1 \times \frac{24}{120} = -300$$

$\therefore \quad y_1 = 0.2\,x_1 - 300$ ...(*iii*)

Also Speed of wheel Y = Speed of wheel Z or shaft I

$\therefore \quad x_1 + y_1 = x + y = -1500$ ...(*iv*)

$x_1 + 0.2\,x_1 - 300 = -1500$ ...[From equation (*iii*)]

$1.2\,x_1 = -1500 + 300 = -1200$

or $\quad x_1 = -1200/1.2 = -1000$

and $\quad y_1 = -1500 - x_1 = -1500 + 1000 = -500$

1. *Speed and direction of the driven shaft O and the wheel P*

Speed of the driven shaft O,

$$N_O = y_1 = -500 = 500 \text{ r.p.m. clockwise } \textbf{Ans.}$$

and Speed of the wheel P, $N_P = y_1 + x_1 \times \frac{T_Y}{T_Q} \times \frac{T_X}{T_P} = -500 - 1000 \times \frac{24}{120} \times \frac{36}{144}$

$$= -550 = 550 \text{ r.p.m. clockwise } \textbf{Ans.}$$

2. *Torque tending to rotate the fixed wheel R*

We know that the torque on shaft I or input torque

$$T_1 = \frac{P \times 60}{2\pi \times N_1} = \frac{7500 \times 60}{2\pi \times 1500} = 47.74 \text{ N-m}$$

and torque on shaft O or output torque,

$$T_2 = \frac{\eta \times P \times 60}{2\pi \times N_O} = \frac{0.8 \times 7500 \times 60}{2\pi \times 500} = 114.58 \text{ N-m}$$

Since the input and output shafts rotate in the same direction (*i.e.* clockwise), therefore input and output torques will be in opposite direction.

$\therefore$ Torque tending to rotate the fixed wheel R

$$= T_2 - T_1 = 114.58 - 47.74 = 66.84 \text{ N-m} \quad \textbf{Ans.}$$

Example 13.24. *An epicyclic bevel gear train (known as Humpage's reduction gear) is shown in Fig. 13.31. It consists of a fixed wheel C, the driving shaft X and the driven shaft Y. The compound wheel B-D can revolve on a spindle F which can turn freely about the axis X and Y.*

Show that **(i)** *if the ratio of tooth numbers* T_B / T_D *is greater than* T_C / T_E*, the wheel E will rotate in the same direction as wheel A, and* **(ii)** *if the ratio* T_B / T_D *is less than* T_C / T_E*, the direction of E is reversed.*

If the numbers of teeth on wheels A, B, C, D and E are 34, 120, 150, 38 and 50 respectively and 7.5 kW is put into the shaft X at 500 r.p.m., what is the output torque of the shaft Y, and what are the forces (tangential to the pitch cones) at the contact points between wheels D and E and between wheels B and C, if the module of all wheels is 3.5 mm ?

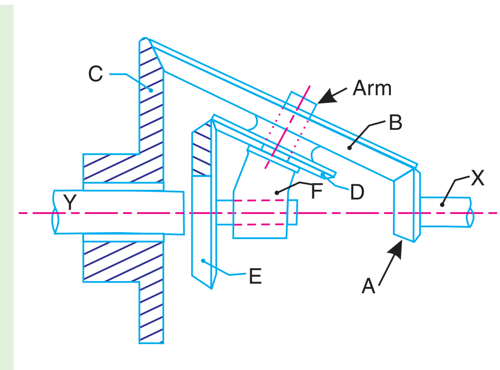


Fig. 13.31

Solution. Given : $T_A = 34$; $T_B = 120$; $T_C = 150$; $T_D = 38$; $T_E = 50$; $P_X = 7.5$ kW = 7500 W ; $N_X = 500$ r.p.m. ; $m = 3.5$ mm

The table of motions is given below :

Table 13.27. Table of motions.

Step No.	Conditions of motion	Revolutions of elements				
		Spindle F	Wheel A (or shaft X)	Compound wheel B-D	Wheel C	Wheel E (or shaft Y)
1.	Spindle fixed, wheel A is rotated through + 1 revolution	0	+ 1	$+\frac{T_A}{T_B}$	$-\frac{T_A}{T_B}\times\frac{T_B}{T_C} = -\frac{T_A}{T_C}$	$-\frac{T_A}{T_B}\times\frac{T_D}{T_E}$
2.	Spindle fixed, wheel A is rotated through $+x$ revolutions	0	$+x$	$+x\times\frac{T_A}{T_B}$	$-x\times\frac{T_A}{T_C}$	$-x\times\frac{T_A}{T_B}\times\frac{T_D}{T_E}$
3.	Add $+y$ revolutions to all elements	$+y$	$+y$	$+y$	$+y$	$+y$
4.	Total motion	$+y$	$x+y$	$y+x\times\frac{T_A}{T_B}$	$y-x\times\frac{T_A}{T_C}$	$y-x\times\frac{T_A}{T_B}\times\frac{T_D}{T_E}$

Let us assume that the driving shaft X rotates through 1 revolution anticlockwise, therefore the wheel A will also rotate through 1 revolution anticlockwise.

$$\therefore \qquad x + y = +1 \quad \text{or} \quad y = 1 - x \qquad \text{...}(i)$$

We also know that the wheel C is fixed, therefore

$$y - x\times\frac{T_A}{T_C} = 0 \quad \text{or} \quad (1-x) - x\times\frac{T_A}{T_C} = 0 \qquad \text{...[From equation } (i)\text{]}$$

$$1 - x\left(1+\frac{T_A}{T_C}\right) = 0 \quad \text{or} \quad x\left(\frac{T_C+T_A}{T_C}\right) = 1$$

and

$$x = \frac{T_C}{T_C+T_A} \qquad \text{...}(ii)$$

From equation (*i*),

$$y = 1 - x = 1 - \frac{T_C}{T_C+T_A} = \frac{T_A}{T_C+T_A} \qquad \text{...}(iii)$$

We know that speed of wheel E,

$$N_E = y - x\times\frac{T_A}{T_B}\times\frac{T_D}{T_E} = \frac{T_A}{T_C+T_A} - \frac{T_C}{T_C+T_A}\times\frac{T_A}{T_B}\times\frac{T_D}{T_E}$$

$$= \frac{T_A}{T_C+T_A}\left(1-\frac{T_C}{T_B}\times\frac{T_D}{T_E}\right) \qquad \text{...}(iv)$$

and the speed of wheel A,

$$N_A = x + y = +1 \text{ revolution}$$

(*i*) If $\frac{T_B}{T_D} > \frac{T_C}{T_E}$ or $T_B\times T_E > T_C\times T_D$, then the equation (*iv*) will be positive. Therefore the wheel E will rotate in the same direction as wheel A. **Ans.**

(ii) If $\frac{T_B}{T_D} < \frac{T_C}{T_E}$ or $T_B \times T_E < T_C \times T_D$, then the equation ***(iv)*** will be negative. Therefore the wheel *E* will rotate in the opposite direction as wheel *A*. **Ans.**

Output torque of shaft Y

We know that the speed of the driving shaft *X* (or wheel *A*) or input speed is 500 r.p.m., therefore from the fourth row of the table,

$$x + y = 500 \quad \text{or} \quad y = 500 - x \qquad \text{...}(v)$$

Since the wheel *C* is fixed, therefore

$$y - x \times \frac{T_A}{T_C} = 0 \quad \text{or} \quad (500 - x) - x \times \frac{34}{150} = 0 \qquad \text{...[From equation } (v)]$$

$$\therefore \quad 500 - x - 0.227\,x = 0 \quad \text{or} \quad x = 500/1.227 = 407.5 \text{ r.p.m.}$$

and
$$y = 500 - x = 500 - 407.5 = 92.5 \text{ r.p.m.}$$

Since the speed of the driven or output shaft *Y* (*i.e.* N_Y) is equal to the speed of wheel *E* (*i.e.* N_E), therefore

$$N_Y = N_E = y - x \times \frac{T_A}{T_B} \times \frac{T_D}{T_E} = 92.5 - 407.5 \times \frac{34}{120} \times \frac{38}{50}$$
$$= 92.5 - 87.75 = 4.75 \text{ r.p.m.}$$

Assuming 100 per cent efficiency of the gear train, input power P_X is equal to output power (P_Y), *i.e.*

$$P_Y = P_X = 7.5 \text{ kW} = 7500 \text{ W}$$

∴ Output torque of shaft *Y*,

$$= \frac{P_Y \times 60}{2\pi\, N_Y} = \frac{7500 \times 60}{2\pi \times 4.75} = 15\,076 \text{ N-m} = 15.076 \text{ kN-m} \quad \textbf{Ans.}$$

Tangential force between wheels D and E

We know that the pitch circle radius of wheel *E*,

$$r_E = \frac{m \times T_E}{2} = \frac{3.5 \times 50}{2} = 87.5 \text{ mm} = 0.0875 \text{ m}$$

∴ Tangential force between wheels *D* and *E*,

$$= \frac{\text{Torque on wheel } E}{\text{Pitch circle radius of wheel } E} = \frac{15.076}{0.0875} = 172.3 \text{ kN} \quad \textbf{Ans.}$$

...(∵ Torque on wheel *E* = Torque on shaft *Y*)

Tangential force between wheels B and C

We know that the input torque on shaft *X* or on wheel *A*

$$= \frac{P_X \times 60}{2\pi\, N_X} = \frac{7500 \times 60}{2\pi \times 500} = 143 \text{ N-m}$$

∴ Fixing torque on the fixed wheel *C*

$$= \text{Torque on wheel } E - \text{Torque on wheel } A$$
$$= 15\,076 - 143 = 14\,933 \text{ N-m} = 14.933 \text{ kN-m}$$

Pitch circle radius of wheel C,

$$r_C = \frac{m \times T_C}{2} = \frac{3.5 \times 150}{2} = 262.5 \text{ mm} = 0.2625 \text{ m}$$

Tangential force between wheels B and C

$$= \frac{\text{Fixing torque on wheel } C}{r_C} = \frac{14.933}{0.2625} = 57 \text{ kN } \textbf{Ans.}$$

EXERCISES

1. A compound train consists of six gears. The number of teeth on the gears are as follows :

Gear :	A	B	C	D	E	F
No. of teeth :	60	40	50	25	30	24

The gears B and C are on one shaft while the gears D and E are on another shaft. The gear A drives gear B, gear C drives gear D and gear E drives gear F. If the gear A transmits 1.5 kW at 100 r.p.m. and the gear train has an efficiency of 80 per cent, find the torque on gear F. **[Ans. 30.55 N-m]**

2. Two parallel shafts are to be connected by spur gearing. The approximate distance between the shafts is 600 mm. If one shaft runs at 120 r.p.m. and the other at 360 r.p.m., find the number of teeth on each wheel, if the module is 8 mm. Also determine the exact distance apart of the shafts. **[Ans. 114, 38 ; 608 mm]**

3. In a reverted gear train, as shown in Fig. 13.32, two shafts A and B are in the same straight line and are geared together through an intermediate parallel shaft C. The gears connecting the shafts A and C have a module of 2 mm and those connecting the shafts C and B have a module of 4.5 mm. The speed of shaft A is to be about but greater than 12 times the speed of shaft B, and the ratio at each reduction is same. Find suitable number of teeth for gears. The number of teeth of each gear is to be a minimum but not less than 16. Also find the exact velocity ratio and the distance of shaft C from A and B. **[Ans. 36, 126, 16, 56 ; 12.25 ; 162 mm]**

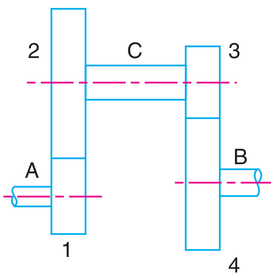


Fig. 13.32

4. In an epicyclic gear train, as shown in Fig.13.33, the number of teeth on wheels A, B and C are 48, 24 and 50 respectively. If the arm rotates at 400 r.p.m., clockwise, find : 1. Speed of wheel C when A is fixed, and 2. Speed of wheel A when C is fixed. **[Ans. 16 r.p.m. (clockwise) ; 16.67 (anticlockwise)]**

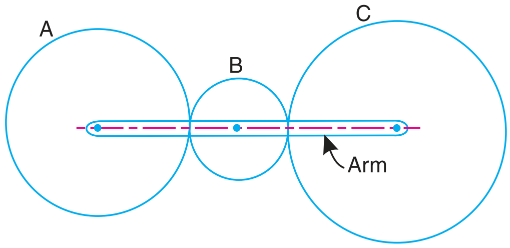


Fig. 13.33

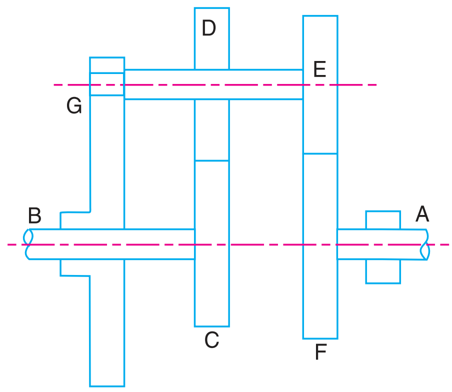


Fig. 13.34

5. In an epicyclic gear train, as shown in Fig. 13.34, the wheel *C* is keyed to the shaft *B* and wheel *F* is keyed to shaft *A*. The wheels *D* and *E* rotate together on a pin fixed to the arm *G*. The number of teeth on wheels *C*, *D*, *E* and *F* are 35, 65, 32 and 68 respectively.

If the shaft *A* rotates at 60 r.p.m. and the shaft *B* rotates at 28 r.p.m. in the opposite direction, find the speed and direction of rotation of arm *G*. **[Ans. 90 r.p.m., in the same direction as shaft *A*]**

6. An epicyclic gear train, as shown in Fig. 13.35, is composed of a fixed annular wheel *A* having 150 teeth. The wheel *A* is meshing with wheel *B* which drives wheel *D* through an idle wheel *C*, *D* being concentric with *A*. The wheels *B* and *C* are carried on an arm which revolves clockwise at 100 r.p.m. about the axis of *A* and *D*. If the wheels *B* and *D* have 25 teeth and 40 teeth respectively, find the number of teeth on *C* and the speed and sense of rotation of *C*. **[Ans. 30 ; 600 r.p.m. clockwise]**

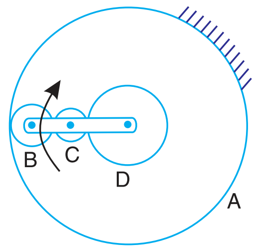


Fig. 13.35

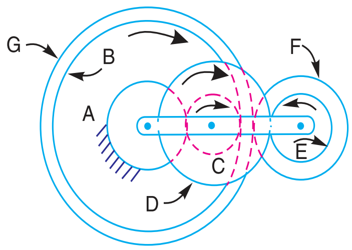


Fig. 13.36

7. Fig. 13.36, shows an epicyclic gear train with the following details :

A has 40 teeth external (fixed gear) ; *B* has 80 teeth internal ; *C* - *D* is a compound wheel having 20 and 50 teeth (external) respectively, *E*-*F* is a compound wheel having 20 and 40 teeth (external) respectively, and *G* has 90 teeth (external).

The arm runs at 100 r.p.m. in clockwise direction. Determine the speeds for gears *C*, *E*, and *B*.

[Ans. 300 r.p.m. clockwise ; 400 r.p.m. anticlockwise ; 150 r.p.m. clockwise]

8. An epicyclic gear train, as shown in Fig. 13.37, has a sun wheel *S* of 30 teeth and two planet wheels *P*-*P* of 50 teeth. The planet wheels mesh with the internal teeth of a fixed annulus *A*. The driving shaft carrying the sunwheel, transmits 4 kW at 300 r.p.m. The driven shaft is connected to an arm which carries the planet wheels. Determine the speed of the driven shaft and the torque transmitted, if the overall efficiency is 95%. **[Ans. 56.3 r.p.m. ; 644.5 N-m]**

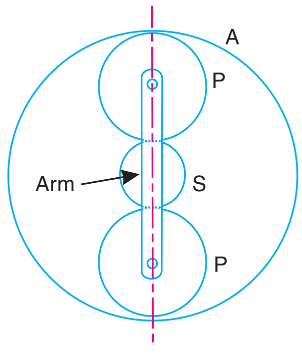


Fig. 13.37

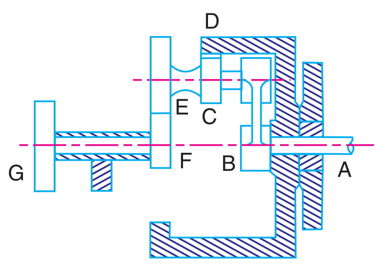


Fig. 13.38

9. An epicyclic reduction gear, as shown in Fig. 13.38, has a shaft *A* fixed to arm *B*. The arm *B* has a pin fixed to its outer end and two gears *C* and *E* which are rigidly fixed, revolve on this pin. Gear *C* meshes with annular wheel *D* and gear *E* with pinion *F*. *G* is the driver pulley and *D* is kept stationary.

The number of teeth are : $D = 80$; $C = 10$; $E = 24$ and $F = 18$.

If the pulley *G* runs at 200 r.p.m. ; find the speed of shaft *A*.

[Ans. 17.14 r.p.m. in the same direction as that of *G*]

10. A reverted epicyclic gear train for a hoist block is shown in Fig. 13.39. The arm *E* is keyed to the same shaft as the load drum and the wheel *A* is keyed to a second shaft which carries a chain wheel, the chain being operated by hand. The two shafts have common axis but can rotate independently. The wheels *B* and *C* are compound and rotate together on a pin carried at the end of arm *E*. The wheel *D* has internal teeth and is fixed to the outer casing of the block so that it does not rotate.

The wheels *A* and *B* have 16 and 36 teeth respectively with a module of 3 mm. The wheels *C* and *D* have a module of 4 mm. Find : 1. the number of teeth on wheels *C* and *D* when the speed of *A* is ten times the speed of arm *E*, both rotating in the same sense, and 2. the speed of wheel *D* when the wheel *A* is fixed and the arm *E* rotates at 450 r.p.m. anticlockwise.

[Ans. T_C = 13 ; T_D = 52 ; 500 r.p.m. anticlockwise]

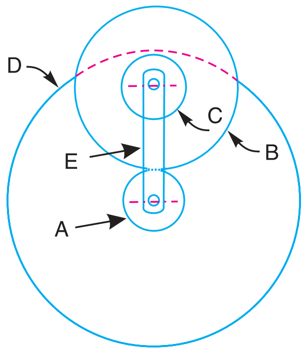


Fig. 13.39

11. A compound epicyclic gear is shown diagrammatically in Fig. 13.40. The gears *A*, *D* and *E* are free to rotate on the axis *P*. The compound gear *B* and *C* rotate together on the axis *Q* at the end of arm *F*. All the gears have equal pitch. The number of external teeth on the gears *A*, *B* and *C* are 18, 45 and 21 respectively. The gears *D* and *E* are annular gears. The gear *A* rotates at 100 r.p.m. in the anticlockwise direction and the gear *D* rotates at 450 r.p.m. clockwise. Find the speed and direction of the arm and the gear *E*. **[Ans. 400 r.p.m. clockwise ; 483.3 r.p.m. clockwise]**

12. In an epicyclic gear train of the 'sun and planet type' as shown in Fig. 13.41, the pitch circle diameter of the internally toothed ring *D* is to be 216 mm and the module 4 mm. When the ring *D* is stationary, the spider *A*, which carries three planet wheels *C* of equal size, is to make one revolution in the same sense as the sun wheel *B* for every five revolutions of the driving spindle carrying the sunwheel *B*. Determine suitable number of teeth for all the wheels and the exact diameter of pitch circle of the ring. **[Ans. T_B = 14 , T_C = 21 , T_D = 56 ; 224 mm]**

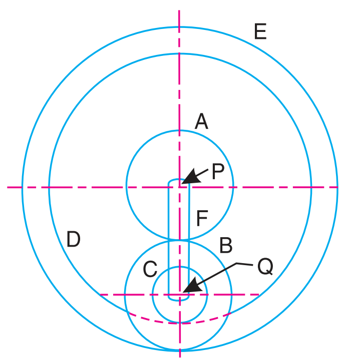


Fig. 13.40

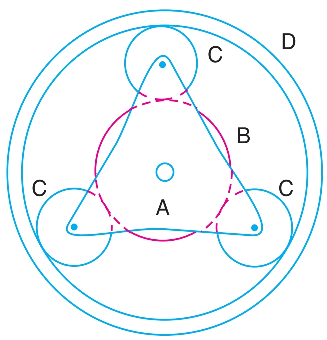


Fig. 13.41

13. An epicyclic train is shown in Fig. 13.42. Internal gear *A* is keyed to the driving shaft and has 30 teeth. Compound wheel *C* and *D* of 20 and 22 teeth respectively are free to rotate on the pin fixed to the arm *P* which is rigidly connected to the driven shaft. Internal gear *B* which has 32 teeth is fixed. If the driving shaft runs at 60 r.p.m. clockwise, determine the speed of the driven shaft. What is the direction of rotation of driven shaft with reference to driving shaft? **[Ans. 1980 r.p.m. clockwise]**

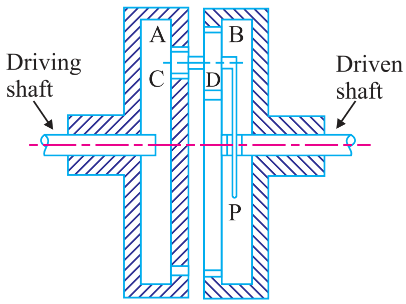


Fig. 13.42

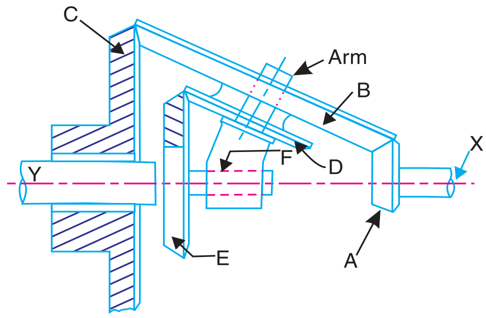


Fig. 13.43

14. A shaft Y is driven by a co-axial shaft X by means of an epicyclic gear train, as shown in Fig. 13.43. The wheel A is keyed to X and E to Y. The wheels B and D are compound and carried on an arm F which can turn freely on the common axes of X and Y. The wheel C is fixed. If the numbers of teeth on A, B, C, D and E are respectively 20, 64, 80, 30 and 50 and the shaft X makes 600 r.p.m., determine the speed in r.p.m. and sense of rotation of the shaft Y.

[Ans. 30 r.p.m. in the same sense as shaft *X*]

15. An epicyclic bevel gear train, as shown in Fig. 13.44, has fixed gear B meshing with pinion C. The gear E on the driven shaft meshes with the pinion D. The pinions C and D are keyed to a shaft, which revolves in bearings on the arm A. The arm A is keyed to the driving shaft. The number of teeth are : $T_B = 75$, $T_C = 20$, $T_D = 18$, and $T_E = 70$. Find the speed of the driven shaft, if 1. the driving shaft makes 1000 r.p.m., and 2. the gear B turns in the same sense as the driving shaft at 400 r.p.m., the driving shaft still making 1000 r.p.m.

[Ans. 421.4 r.p.m. in the same direction as driving shaft]

16. The epicyclic gear train is shown in Fig. 13.45. The wheel D is held stationary by the shaft A and the arm B is rotated at 200 r.p.m. The wheels E (20 teeth) and F (40 teeth) are fixed together and rotate freely on the pin carried by the arm. The wheel G (30 teeth) is rigidly attached to the shaft C. Find the speed of shaft C stating the direction of rotation to that of B.

If the gearing transmits 7.5 kW, what will be the torque required to hold the shaft A stationary, neglecting all friction losses?

[Ans. 466.7 r.p.m. in opposite direction of *B*; 511.5 N-m in opposite direction of *B*]

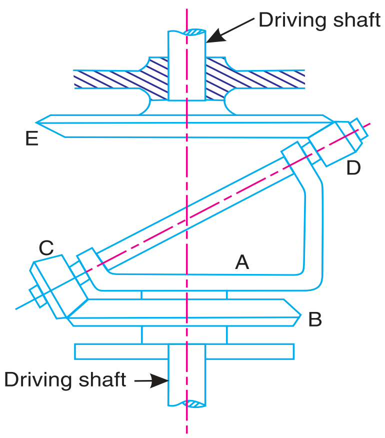


Fig. 13.44

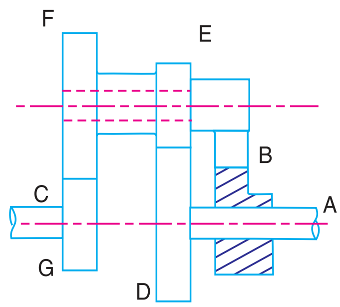


Fig. 13.45

17. An epicyclic gear train, as shown in Fig. 13.46, consists of two sunwheels A and D with 28 and 24 teeth respectively, engaged with a compound planet wheels B and C with 22 and 26 teeth. The sunwheel

D is keyed to the driven shaft and the sunwheel A is a fixed wheel co-axial with the driven shaft. The planet wheels are carried on an arm E from the driving shaft which is co-axial with the driven shaft.

Find the velocity ratio of gear train. If 0.75 kW is transmitted and input speed being 100 r.p.m., determine the torque required to hold the sunwheel A. **[Ans. 2.64 ; 260.6 N-m]**

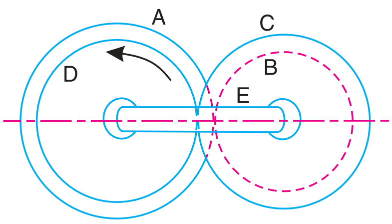


Fig. 13.46

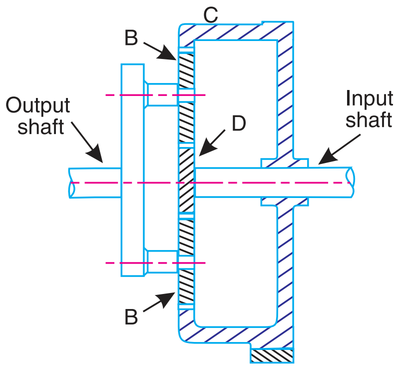


Fig. 13.47

18. In the epicyclic reduction gear, as shown in Fig. 13.47, the sunwheel D has 20 teeth and is keyed to the input shaft. Two planet wheels B, each having 50 teeth, gear with wheel D and are carried by an arm A fixed to the output shaft. The wheels B also mesh with an internal gear C which is fixed. The input shaft rotates at 2100 r.p.m. Determine the speed of the output shaft and the torque required to fix C when the gears are transmitting 30 kW.

[Ans. 300 r.p.m. in the same sense as the input shaft ; 818.8 N-m]

19. An epicyclic gear train for an electric motor is shown in Fig. 13.48. The wheel S has 15 teeth and is fixed to the motor shaft rotating at 1450 r.p.m. The planet P has 45 teeth, gears with fixed annulus A and rotates on a spindle carried by an arm which is fixed to the output shaft. The planet P also gears with the sun wheel S. Find the speed of the output shaft. If the motor is transmitting 1.5 kW, find the torque required to fix the annulus A. **[Ans. 181.3 r.p.m. ; 69.14 N-m]**

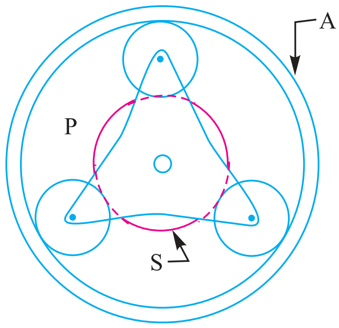


Fig. 13.48

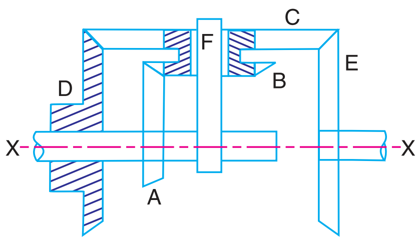


Fig. 13.49

20. An epicyclic gear consists of bevel wheels as shown in Fig. 13.49. The driving pinion A has 20 teeth and meshes with the wheel B which has 25 teeth. The wheels B and C are fixed together and turn freely on the shaft F. The shaft F can rotate freely about the main axis XX. The wheel C has 50 teeth and meshes with wheels D and E, each of which has 60 teeth. Find the speed and direction of E when A rotates at 200 r.p.m., if

1. D is fixed, and **2.** D rotates at 100 r.p.m., in the same direction as A.

In both the cases, find the ratio of the torques transmitted by the shafts of the wheels A and E, the friction being neglected.

[Ans. 800 r.p.m. in the opposite direction of A ; 300 r.p.m. in the opposite direction of A ; 4 ; 1.5]

DO YOU KNOW ?

1. What do you understand by 'gear train'? Discuss the various types of gear trains.
2. Explain briefly the differences between simple, compound, and epicyclic gear trains. What are the special advantages of epicyclic gear trains ?
3. Explain the procedure adopted for designing the spur wheels.
4. How the velocity ratio of epicyclic gear train is obtained by tabular method?
5. Explain with a neat sketch the 'sun and planet wheel'.
6. What are the various types of the torques in an epicyclic gear train ?

OBJECTIVE TYPE QUESTIONS

1. In a simple gear train, if the number of idle gears is odd, then the motion of driven gear will
   (*a*) be same as that of driving gear
   (*b*) be opposite as that of driving gear
   (*c*) depend upon the number of teeth on the driving gear
   (*d*) none of the above
2. The train value of a gear train is
   (*a*) equal to velocity ratio of a gear train
   (*b*) reciprocal of velocity ratio of a gear train
   (*c*) always greater than unity
   (*d*) always less than unity
3. When the axes of first and last gear are co-axial, then gear train is known as
   (*a*) simple gear train
   (*b*) compound gear train
   (*c*) reverted gear train
   (*d*) epicyclic gear train
4. In a clock mechanism, the gear train used to connect minute hand to hour hand, is
   (*a*) epicyclic gear train
   (*b*) reverted gear train
   (*c*) compound gear train
   (*d*) simple gear train
5. In a gear train, when the axes of the shafts, over which the gears are mounted, move relative to a fixed axis, is called
   (*a*) simple gear train
   (*b*) compound gear train
   (*c*) reverted gear train
   (*d*) epicyclic gear train
6. A differential gear in an automobile is a
   (*a*) simple gear train
   (*b*) epicyclic gear train
   (*c*) compound gear train
   (*d*) none of these
7. A differential gear in automobiles is used to
   (*a*) reduce speed
   (*b*) assist in changing speed
   (*c*) provide jerk-free movement of vehicle
   (*d*) help in turning

ANSWERS

1. (*a*)	**2.** (*b*)	**3.** (*c*)	**4.** (*b*)	**5.** (*d*)
6. (*b*)	**7.** (*d*)			

GO To FIRST

of lap on the smaller pulley, the idler pulley is used. The angle of lap may be defined as the angle of contact between the belt and the pulley. With the increase in angle of lap, the belt drive can transmit more power. Along with the increase in angle of lap, the idler pulley also does not allow reduction in the initial tension in the belt. The use of idler pulley is shown in Figure 3.7.

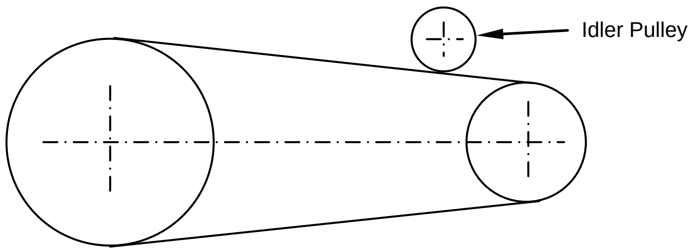


Figure 3.7 : Use of Idler in Belt Drive

SAQ 2

(a) What is the main advantage of idler pulley?

(b) A prime mover drives a dc generator by belt drive. The speeds of prime mover and generator are 300 rpm and 500 rpm, respectively. The diameter of the driver pulley is 600 mm. The slip in the drive is 3%. Determine diameter of the generator pulley if belt is 6 mm thick.

3.4.1 Law of Belting

The law of belting states that the centre line of the belt as it approaches the pulley, must lie in plane perpendicular to the axis of the pulley in the mid plane of the pulley otherwise the belt will run off the pulley. However, the point at which the belt leaves the other pulley must lie in the plane of a pulley.

The Figure 3.8 below shows the belt drive in which two pulleys are at right angle to each other. It can be seen that the centre line of the belt approaching larger or smaller pulley lies in its plane. The point at which the belt leaves is contained in the plane of the other pulley.

If motion of the belt is reversed, the law of the belting will be violated. Therefore, motion is possible in one direction in case of non-parallel shafts as shown in Figure 3.8.

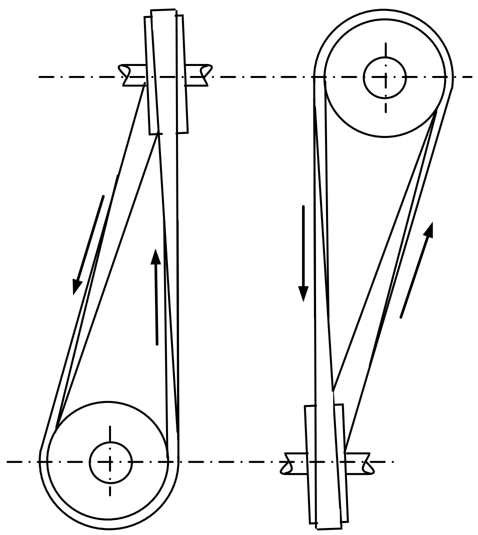


Figure 3.8 : Law of Belting

3.4.2 Length of the Belt

For any type of the belt drive it is always desirable to know the length of belt required. It will be required in the selection of the belt. The length can be determined by the geometric considerations. However, actual length is slightly shorter than the theoretically determined value.

Open Belt Drive

The open belt drive is shown in Figure 3.9. Let O_1 and O_2 be the pulley centers and *AB* and *CD* be the common tangents on the circles representing the two pulleys. The total length of the belt '*L*' is given by

$$L = AB + \text{Arc } BHD + DC + \text{Arc } CGA$$

Let r be the radius of the smaller pulley,

R be the radius of the larger pulley,

C be the centre distance between the pulleys, and

β be the angle subtended by the tangents *AB* and *CD* with $O_1\,O_2$.

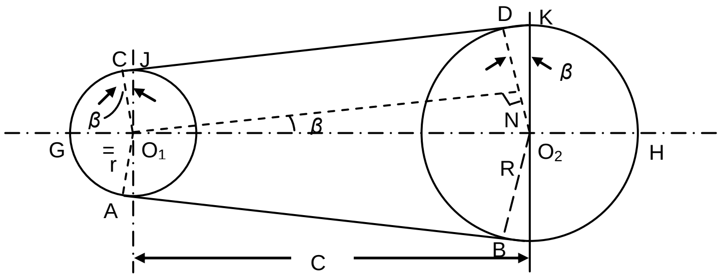


Figure 3.9 : Open Belt Drive

Draw $O_1\,N$ parallel to *CD* to meet $O_2\,D$ at *N*.

By geometry, $\angle\, O_2\,O_1,\, N = \angle\, C\,O_1\,J = \angle\, D\,O_2\,K = \beta$

$$\text{Arc } BHD = (\pi + 2\beta)\, R,$$

$$\text{Arc } CGA = (\pi - 2\beta)\, r$$

$$AB = CD = O_1\,N = O_1\,O_2 \cos\beta = C \cos\beta$$

$$\sin\beta = \frac{R - r}{C}$$

or, $$\beta = \sin^{-1}\frac{(R - r)}{C}$$

$$\cos\beta = \sqrt{1 - \sin^2\beta} \approx \left(1 - \frac{1}{2}\sin^2\beta\right)$$

$\therefore$ $$L = (\pi + 2\beta)\, R + (\pi - 2\beta)\, r + 2C\left(1 - \frac{1}{2}\sin^2\beta\right)$$

For small value of β; $\beta = \dfrac{(R - r)}{C}$, the approximate lengths

$$L = \pi\,(R + r) + 2\,(R - r)\,\frac{(R - r)}{C} + 2C\left[1 - \frac{1}{2}\left(\frac{R - r}{C}\right)^2\right]$$

$$= \pi\,(R + r) + \frac{(R - r)^2}{C} + 2C\left[1 - \frac{1}{2}\left(\frac{R - r}{C}\right)^2\right]$$

This provides approximate length because of the approximation taken earlier.

Crossed-Belt Drive

The crossed-belt drive is shown in Figure 3.10. Draw $O_1 N$ parallel to the line CD which meets extended $O_2 D$ at N. By geometry

$$\angle C O_1 J = \angle D O_2 K = \angle O_2 O_1 N$$

$$L = \text{Arc } AGC + AB + \text{Arc } BKD + CD$$

$$\text{Arc } AGC = r\ (\pi + 2\beta), \text{ and Arc } BKD = (\pi + 2\beta)\ R$$

$$\sin \beta = \frac{R + r}{C} \quad \text{or} \quad \beta = \sin^{-1} \frac{(R + r)}{C}$$

For small value of β

$$\beta \approx \frac{R + r}{C}$$

$$\cos \beta = \sqrt{1 + \sin^2 \beta} \approx \left(1 - \frac{1}{2} \sin^2 \beta\right) = \left[1 - \frac{1}{2} \frac{(R + r)^2}{C^2}\right]$$

$$L = r\ (\pi + 2\beta) + 2C \cos \beta + R\ (\pi + 2\beta)$$

$$= (\pi + 2\beta)\ (R + r) + 2C \cos \beta$$

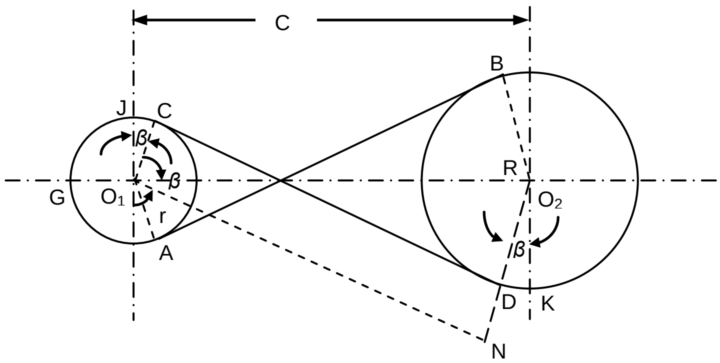


Figure 3.10 : Cross Belt Drive

For approximate length

$$L = \pi\ (R + r) + 2\ \frac{(R + r)^2}{C} + 2C \left[1 - \frac{1}{2} \frac{(R + r)^2}{C^2}\right]$$

$$= \pi\ (R + r) + \frac{(R + r)^2}{C} + 2C$$

SAQ 3

Which type of drive requires longer length for same centre distance and size of pulleys?

3.4.3 Cone Pulleys

Sometimes the driving shaft is driven by the motor which rotates at constant speed but the driven shaft is designed to be driven at different speeds. This can be easily done by using stepped or cone pulleys as shown in Figure 3.11. The cone pulley has different sets of radii and they are selected such that the same belt can be used at different sets of the cone pulleys.

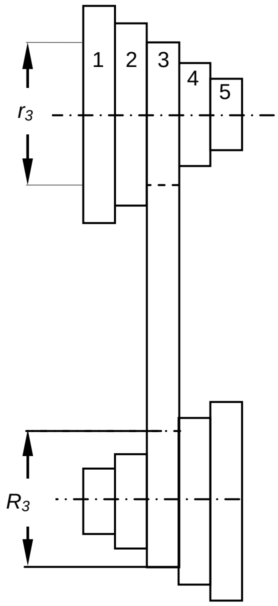


Figure 3.11 : Cone Pulleys

Let N_d be the speed of the driving shaft which is constant.

N_n be the speed of the driven shaft when the belt is on nth step.

r_n be the radius of the nth step of driving pulley.

R_n be the radius of the nth step of the driven pulley.

where n is an integer, 1, 2, . . .

The speed ratio is inversely proportional to the pulley radii

$$\therefore \qquad \frac{N_1}{N_d} = \frac{r_1}{R_1} \qquad \ldots (3.1)$$

For this first step radii r_1 and R_1 can be chosen conveniently.

For second pair $\frac{N_2}{N_d} = \frac{r_2}{R_2}$, and similarly $\frac{N_n}{N_d} = \frac{r_n}{R_n}$.

In order to use same belt on all the steps, the length of the belt should be same

$$\text{i.e.} \qquad L_1 = L_2 = \ldots = L_n \qquad \ldots (3.2)$$

Thus, two equations are available – one provided by the speed ratio and other provided by the length relation and for selected speed ratio, the two radii can be calculated. Also it has to be kept in mind that the two pulleys are same. It is desirable that the speed ratios should be in geometric progression.

Let k be the ratio of progression of speed.

$$\therefore \qquad \frac{N_2}{N_1} = \frac{N_3}{N_2} = \ldots \frac{N_n}{N_{n-1}} = k$$

$$\therefore \qquad N_2 = k\, N_1 \text{ and } N_3 = k^2\, N_1$$

$$\therefore \qquad N_n = k^{n-1}\, N_1 = k^{n-1}\, N_d\, \frac{r_1}{R_1}$$

$$\therefore \qquad \frac{r_2}{R_2} = k\, \frac{r_1}{R_1} \text{ and } \frac{r_3}{R_3} = k^2\, \frac{r_1}{R_1}$$

Since, both the pulleys are made similar.

$$\frac{r_n}{R_n} = \frac{R_1}{r_1} \text{ or } k^{n-1}\frac{r_1}{R_1} = \frac{R_1}{r_1}$$

or, $$\frac{R_1}{r_1} = \sqrt{k^{n-1}} \qquad \ldots (3.3)$$

If radii R_1 and r_1 have been chosen, the above equations provides value of k or vice-versa.

SAQ 4

How the speed ratios are selected for cone pulleys?

3.4.4 Ratio of Tensions

The belt drive is used to transmit power from one shaft to the another. Due to the friction between the pulley and the belt one side of the belt becomes tight side and other becomes slack side. We have to first determine ratio of tensions.

Flat Belt

Let tension on the tight side be 'T_1' and the tension on the slack side be 'T_2'. Let 'θ' be the angle of lap and let 'μ' be the coefficient of friction between the belt and the pulley. Consider an infinitesimal length of the belt *PQ* which subtend an angle $\delta\theta$ at the centre of the pulley. Let '*R*' be the reaction between the element and the pulley. Let '*T*' be tension on the slack side of the element, i.e. at point *P* and let '$(T + \delta T)$' be the tension on the tight side of the element.

The tensions *T* and $(T + \delta T)$ shall be acting tangential to the pulley and thereby normal to the radii *OP* and *OQ*. The friction force shall be equal to 'μR' and its action will be to prevent slipping of the belt. The friction force will act tangentially to the pulley at the point *S*.

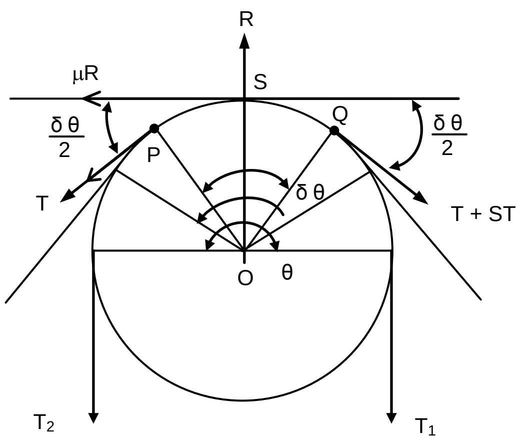


Figure 3.12 : Ratio of Tensions in Flat Belt

Considering equilibrium of the element at *S* and equating it to zero.

Resolving all the forces in the tangential direction

$$\mu R + T\cos\frac{\delta\theta}{2} - (T + \delta T)\cos\frac{\delta\theta}{2} = 0$$

or, $$\mu R = \delta T\cos\frac{\delta\theta}{2} \qquad \ldots (3.4)$$

Resolving all the forces in the radial direction at S and equating it to zero.

$$R - T \sin \frac{\delta\theta}{2} - (T + \delta T) \sin \frac{\delta\theta}{2} = 0$$

or, $$R = (2T + \delta T) \sin \frac{\delta\theta}{2}$$

Since $\delta\theta$ is very small, taking limits

$\therefore$ $$\cos \frac{\delta\theta}{2} \approx 1 \text{ and } \sin \frac{\delta\theta}{2} = \frac{\delta\theta}{2}$$

$\therefore$ $$R = (2T + \delta T) \frac{\delta\theta}{2} = T\,\delta\theta + \delta T \frac{\delta\theta}{2}$$

Neglecting the product of the two infinitesimal quantities $\left(\delta T \dfrac{\delta\theta}{2}\right)$ which is negligible in comparison to other quantities :

$\therefore$ $$R \approx T\,\delta\theta$$

Substituting the value of R and $\cos \dfrac{\delta\theta}{2} \approx 1$ in Eq. (3.4), we get

$$\mu T\,\delta\theta = \delta T$$

or, $$\frac{\delta T}{T} = \mu\,\delta\theta$$

Taking limits on both sides as $\delta\,\theta \to 0$

$$\frac{dT}{T} = \mu\,d\,\theta$$

Integrating between limits, it becomes

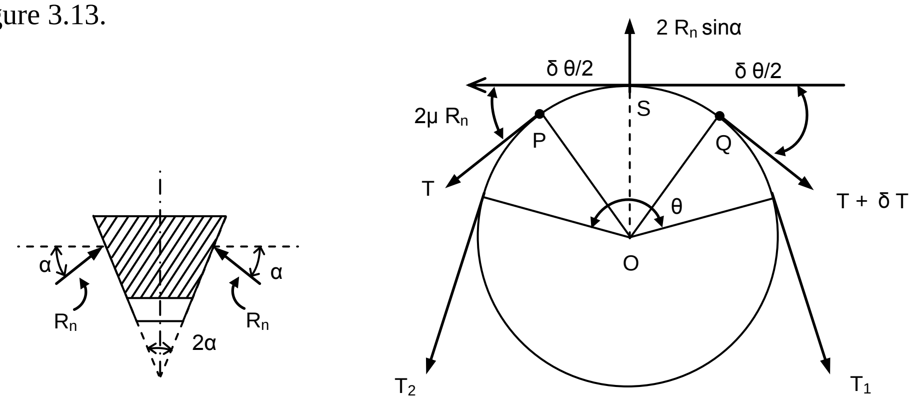
$$\int_{T_2}^{T_1} \frac{dT}{T} = \int_0^{\theta} \mu\,d\theta$$

or, $$\ln \frac{T_1}{T_2} = \mu\theta$$

or, $$\frac{T_1}{T_2} = e^{\mu\theta} \qquad \text{. . . (3.5)}$$

V-belt or Rope

The V-belt or rope makes contact on the two sides of the groove as shown in Figure 3.13.



(a) **(b)**

Figure 3.13 : Ratio of Tension in V-Belt

Let the reaction be 'R_n' on each of the two sides of the groove. The resultant reaction will be $2R_n \sin \alpha$ at point S.

Resolving all the forces tangentially in the Figure 3.13(b), we get

$$2\mu R_n + T \cos \frac{\delta\theta}{2} - (T + \delta T) \cos \frac{\delta\theta}{2} = 0$$

or, $$2\mu R_n = \delta T \cos \frac{\delta\theta}{2} \qquad \text{. . . (3.6)}$$

Resolving all the forces radially, we get

$$2R_n \sin \alpha = T \sin \frac{\delta\theta}{2} + (T + \delta T) \sin \frac{\delta\theta}{2}$$

$$= (2T + \delta T) \sin \frac{\delta\theta}{2}$$

Since $\delta\theta$ is very small

$$\sin \frac{\delta\theta}{2} \approx \frac{\delta\theta}{2}$$

$\therefore$ $$2R_n \sin \alpha = (2T + \delta T) \frac{\delta\theta}{2} = T\,\delta\theta + \delta T \times \frac{\delta\theta}{2}$$

Neglecting the product of the two infinitesimal quantities

$$2R_n \sin \alpha \approx T\,\delta\theta$$

or, $$R_n \approx \frac{T\,\delta\theta}{2 \sin \alpha}$$

Substituting the value of R_n and using the approximation $\cos \frac{\delta\theta}{2} \approx 1$, in Eq. (3.6), we get

$$\mu \frac{T\,\delta\theta}{\sin \alpha} = \delta T$$

or, $$\frac{T\,\delta\theta}{T} = \frac{\mu}{\sin \alpha}\,\delta\theta$$

Taking the limits and integrating between limits, we get

$$\int_{T_2}^{T_1} \frac{dT}{T} = \int_0^{\theta} \frac{\mu}{\sin \alpha}\,d\theta$$

or, $$\ln \frac{T_1}{T_2} = \frac{\mu}{\sin \alpha}\,\theta$$

or, $$\frac{T_1}{T_2} = e^{\frac{\mu}{\sin \alpha}\theta} \qquad \text{. . . (3.7)}$$

SAQ 5

(a) If a rope makes two full turn and one quarter turn how much will be angle of lap?

(b) If smaller pulley has coefficient of friction 0.3 and larger pulley has coefficient of friction 0.2. The angle of lap on smaller and larger pulleys are 160^o and 200^o which value of ($\mu\theta$) should be used for ratio of tensions?

3.4.5 Power Transmitted by Belt Drive

The power transmitted by the belt depends on the tension on the two sides and the belt speed.

Let T_1 be the tension on the tight side in 'N'

T_2 be the tension on the slack side in 'N', and

V be the speed of the belt in m/sec.

Then power transmitted by the belt is given by

$$\text{Power } P = (T_1 - T_2)\, V \text{ Watt}$$

$$= \frac{(T_1 - T_2)\, V}{1000} \text{ kW} \qquad \ldots (3.8)$$

or,
$$P = \frac{T_1 \left(1 - \frac{T_2}{T_1}\right) V}{1000} \text{ kW}$$

If belt is on the point of slipping.

$$\frac{T_1}{T_2} = e^{\mu\theta}$$

$$\therefore \qquad P = \frac{T_1\,(1 - e^{-\mu\theta})\, V}{1000} \text{ kW} \qquad \ldots (3.9)$$

The maximum tension T_1 depends on the capacity of the belt to withstand force. If allowable stress in the belt is 'σ_t' in 'Pa', i.e. N/m^2, then

$$T_1 = (\sigma_t \times t \times b) \text{ N} \qquad \ldots (3.10)$$

where t is thickness of the belt in 'm' and b is width of the belt also in m.

The above equations can also be used to determine 'b' for given power and speed.

3.4.6 Tension due to Centrifugal Forces

The belt has mass and as it rotates along with the pulley it is subjected to centrifugal forces. If we assume that no power is being transmitted and pulleys are rotating, the centrifugal force will tend to pull the belt as shown in Figure 3.14(b) and, thereby, a tension in the belt called centrifugal tension will be introduced.

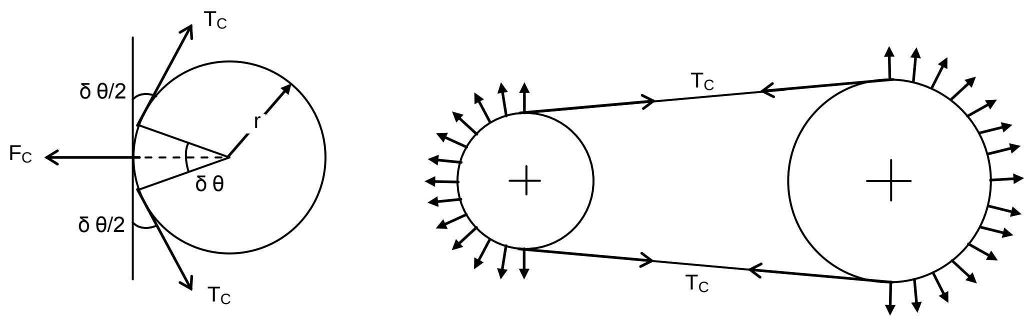


Figure 3.14 : Tension due to Centrifugal Foces

Let 'T_C' be the centrifugal tension due to centrifugal force.

Let us consider a small element which subtends an angle $\delta\theta$ at the centre of the pulley.

Let 'm' be the mass of the belt per unit length of the belt in 'kg/m'.

The centrifugal force 'F_c' on the element will be given by

$$F_C = (r\ \delta\theta\ m) \times \frac{V^2}{r}$$

where V is speed of the belt in m/sec. and r is the radius of pulley in 'm'.

Resolving the forces on the element normal to the tangent

$$F_C - 2T_C \sin \frac{\delta\theta}{2} = 0$$

Since $\delta\theta$ is very small.

$\therefore$ $$\sin \frac{\delta\theta}{2} \approx \frac{\delta\theta}{2}$$

or, $$F_C - 2T_C \frac{\delta\theta}{2} = 0$$

or, $$F_C = T_C\ \delta\theta$$

Substituting for F_C

$$\frac{m\,V^2}{r}\ r\ \delta\theta = T_C\ \delta\theta$$

or, $$T_C = m\,V^2 \qquad \ldots (3.11)$$

Therefore, considering the effect of the centrifugal tension, the belt tension on the tight side when power is transmitted is given by

Tension of tight side $T_t = T_1 + T_C$ and tension on the slack side $T_s = T_2 + T_C$.

The centrifugal tension has an effect on the power transmitted because maximum tension can be only T_t which is

$$T_t = \sigma_t \times t \times b$$

$\therefore$ $$T_1 = \sigma_t \times t \times b - m\,V^2$$

SAQ 6

What will be the centrifugal tension if mass of belt is zero?

3.4.7 Initial Tension

When a belt is mounted on the pulley some amount of initial tension say 'T_0' is provided in the belt, otherwise power transmission is not possible because a loose belt cannot sustain difference in the tension and no power can be transmitted.

When the drive is stationary the total tension on both sides will be '2 T_0'.

When belt drive is transmitting power the total tension on both sides will be (T_1 + T_2), where T_1 is tension on tight side, and T_2 is tension on the slack side.

If effect of centrifugal tension is neglected.

$$2T_0 = T_1 + T_2$$

or, $$T_0 = \frac{T_1 + T_2}{2}$$

If effect of centrifugal tension is considered, then

$$T_0 = T_t + T_s = T_1 + T_2 + 2T_C$$

or, $$T_0 = \frac{T_1 + T_2}{2} + T_C \qquad \ldots (3.12)$$

3.4.8 Maximum Power Transmitted

The power transmitted depends on the tension 'T_1', angle of lap θ, coefficient of friction 'μ' and belt speed 'V'. For a given belt drive, the maximum tension (T_t), angle of lap and coefficient of friction shall remain constant provided that

(a) the tension on tight side, i.e. maximum tension should be equal to the maximum permissible value for the belt, and

(b) the belt should be on the point of slipping.

Therefore, Power $P = T_1 \, (1 - e^{-\mu\theta})$ V

Since, $T_1 = T_t + T_c$

or, $$P = (T_t - T_c)\,(1 - e^{-\mu\theta})\,V$$

or, $$P = (T_t - m\,V^2)\,(1 - e^{-\mu\theta})\,V$$

For maximum power transmitted

$\therefore$ $$\frac{dP}{dV} = (T_t - 3m\,V^2)\,(1 - e^{-\mu\theta})$$

or, $$T_t - 3m\,V^2 = 0$$

or, $$T_t - 3T_c = 0$$

or, $$T_c = \frac{T_t}{3}$$

or, $$m\,V^2 = \frac{T_t}{3}$$

Also, $$V = \sqrt{\frac{T_t}{3m}} \qquad \ldots (3.13)$$

At the belt speed given by the Eq. (3.13) the power transmitted by the belt drive shall be maximum.

SAQ 7

What is the value of centrifugal tension corresponding to the maximum power transmitted?

3.5 KINEMATICS OF CHAIN DRIVE

The chain is wrapped round the sprocket as shown in Figure 3.4(d). The chain in motion is shown in Figure 3.15. It may be observed that the position of axial line changes between the two position as shown by the dotted line and full line. The dotted line meets at point *B* when extended with the line of centers. The firm line meets the line of centers at point *A* when extended. The speed of the driving sprocket say 'ω_1' shall be constant but the velocity of chain will vary between $\omega_1 \times O_1 C$ and $\omega_1 \times O_1 D$. Therefore,

$$\frac{\omega_2}{\omega_1} = \frac{O_1 A}{O_2 B}$$

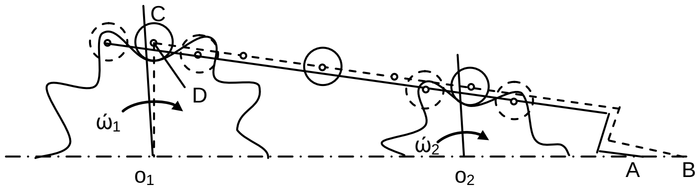


Figure 3.15 : Kinematics of Chain Drive

The variation in the chain speed causes the variation in the angular speed of the driven sprocket. The angular speed of the driven sprocket will vary between

$$\omega_1 \frac{O_1 B}{O_2 B} \quad \text{and} \quad \omega_1 \frac{O_1 A}{O_2 A}$$

This variation can be reduced by increasing number of teeth on the sprocket.

3.6 CLASSIFICATION OF GEARS

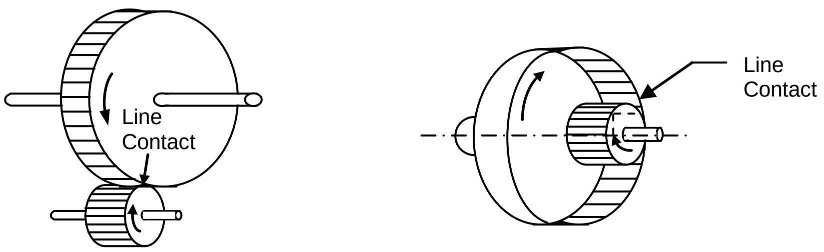
There are different types of arrangement of shafts which are used in practice. According to the relative positions of shaft axes, different types of gears are used.

3.6.1 Parallel Shafts

In this arrangement, the shaft axes lie in parallel planes and remain parallel to one another. The following type of gears are used on these shafts :

Spur Gears

These gears have straight teeth with their alignment parallel to the axes. These gears are shown in mesh in Figures 3.16(a) and (b). The contact between the two meshing teeth is along a line whose length is equal to entire length of teeth. It may be observed that in external meshing, the two shafts rotate opposite to each other whereas in internal meshing the shafts rotate in the same sense.



(a) External Meshing **(b) Internal Meshing**

Figure 3.16 : Spur Gears

If the gears mesh externally and diameter of one gear becomes infinite, the arrangement becomes 'Spur Rack and Pinion'. This is shown in Figure 3.17. It converts rotary motion into translatory motion, or vice-versa.

UNIT 5 GOVERNORS

Structure

5.1 INTRODUCTION

In the last unit, you studied flywheel which minimises fluctuations of speed within the cycle but it cannot minimise fluctuations due to load variation. This means flywheel does not exercise any control over mean speed of the engine. To minimise fluctuations in the mean speed which may occur due to load variation, governor is used. The governor has no influence over cyclic speed fluctuations but it controls the mean speed over a long period during which load on the engine may vary.

When there is change in load, variation in speed also takes place then governor operates a regulatory control and adjusts the fuel supply to maintain the mean speed nearly constant. Therefore, the governor automatically regulates through linkages, the energy supply to the engine as demanded by variation of load so that the engine speed is maintained nearly constant.

Figure 5.1 shows an illustrative sketch of a governor along with linkages which regulates the supply to the engine. The governor shaft is rotated by the engine. If load on the engine increases the engine speed tends to reduce, as a result of which governor balls move inwards. This causes sleeve to move downwards and this movement is transmitted to the valve through linkages to increase the opening and, thereby, to increase the supply.

On the other hand, reduction in the load increases engine speed. As a result of which the governor balls try to fly outwards. This causes an upward movement of the sleeve and it reduces the supply. Thus, the energy input (fuel supply in IC engines, steam in steam turbines, water in hydraulic turbines) is adjusted to the new load on the engine. Thus the governor senses the change in speed and then regulates the supply. Due to this type of action it is simple example of a mechanical feedback control system which senses the output and regulates input accordingly.

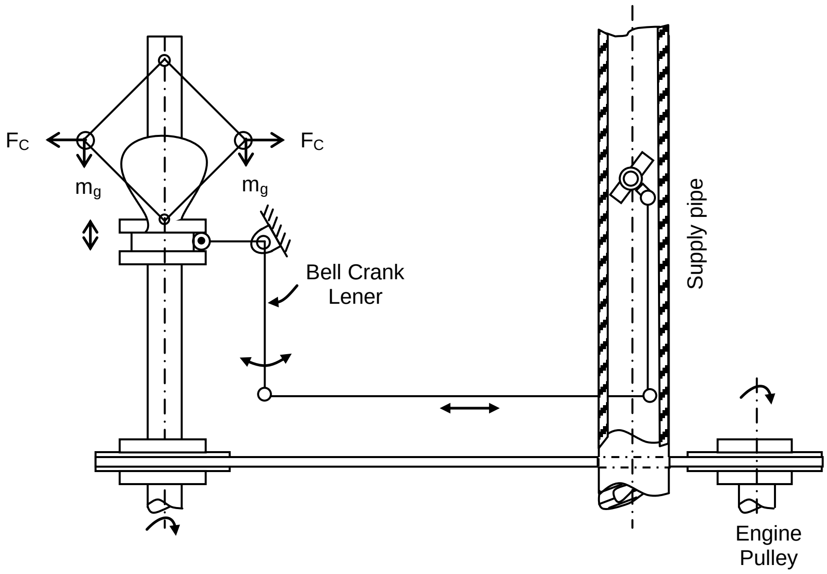


Figure 5.1 : Governor and Linkages

Objectives

After studying this unit, you should be able to

- classify governors,
- analyse different type of governors,
- know characteristics of governors,
- know stability of spring controlled governors, and
- compare different type of governors.

5.2 CLASSIFICATION OF GOVERNORS

The broad classification of governor can be made depending on their operation.

(a) Centrifugal governors

(b) Inertia and flywheel governors

(c) Pickering governors.

Centrifugal Governors

In these governors, the change in centrifugal forces of the rotating masses due to change in the speed of the engine is utilised for movement of the governor sleeve. One of this type of governors is shown in Figure 5.1. These governors are commonly used because of simplicity in operation.

Inertia and Flywheel Governors

In these governors, the inertia forces caused by the angular acceleration of the engine shaft or flywheel by change in speed are utilised for the movement of the balls. The movement of the balls is due to the rate of change of speed in stead of change in speed itself as in case of centrifugal governors. Thus, these governors are more sensitive than centrifugal governors.

Pickering Governors

This type of governor is used for driving a gramophone. As compared to the centrifugal governors, the sleeve movement is very small. It controls the speed by dissipating the excess kinetic energy. It is very simple in construction and can be used for a small machine.

5.2.1 Types of Centrifugal Governors

Depending on the construction these governors are of two types :

(a) Gravity controlled centrifugal governors, and

(b) Spring controlled centrifugal governors.

Gravity Controlled Centrifugal Governors

In this type of governors there is gravity force due to weight on the sleeve or weight of sleeve itself which controls movement of the sleeve. These governors are comparatively larger in size.

Spring Controlled Centrifugal Governors

In these governors, a helical spring or several springs are utilised to control the movement of sleeve or balls. These governors are comparatively smaller in size.

SAQ 1

(a) Compare flywheel with governor.

(b) Which type of control the governor system is?

(c) Compare centrifugal governors with inertia governors.

5.3 GRAVITY CONTROLLED CENTRIFUGAL GOVERNORS

There are three commonly used gravity controlled centrifugal governors :

(a) Watt governor

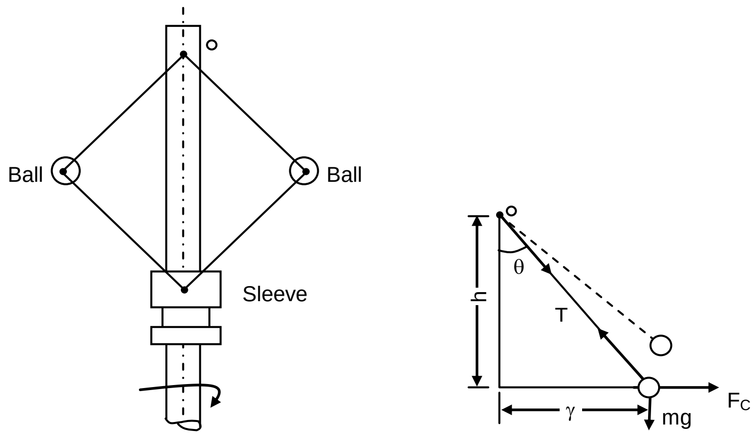
(b) Porter governor

(c) Proell governor

Watt governor does not carry dead weight at the sleeve. Porter governor and proell governor have heavy dead weight at the sleeve. In porter governor balls are placed at the junction of upper and lower arms. In case of proell governor the balls are placed at the extension of lower arms. The sensitiveness of watt governor is poor at high speed and this limits its field of application. Porter governor is more sensitive than watt governor. The proell governor is most sensitive out of these three.

5.3.1 Watt Governor

This governor was used by James Watt in his steam engine. The spindle is driven by the output shaft of the prime mover. The balls are mounted at the junction of the two arms. The upper arms are connected to the spindle and lower arms are connected to the sleeve as shown in Figure 5.2(a).



(a) (b)

Figure 5.2 : Watt Governor

We ignore mass of the sleeve, upper and lower arms for simplicity of analysis. We can ignore the friction also. The ball is subjected to the three forces which are centrifugal force (F_c), weight (mg) and tension by upper arm (T). Taking moment about point O (intersection of arm and spindle axis), we get

$$F_C\ h - mg\ r = 0$$

Since, $$F_C = mr\ \omega^2$$

$\therefore$ $$mr\ \omega^2 h - mg\ r = 0$$

or $$\omega^2 = \frac{g}{h} \qquad \ldots (5.1)$$

$$\omega = \frac{2\pi N}{60}$$

$\therefore$ $$h = \frac{g \times 3600}{4\pi^2\ N^2} = \frac{894.56}{N^2} \qquad \ldots (5.2)$$

where 'N' is in rpm.

Figure 5.3 shows a graph between height 'h' and speed 'N' in rpm. At high speed the change in height h is very small which indicates that the sensitiveness of the governor is very poor at high speeds because of flatness of the curve at higher speeds.

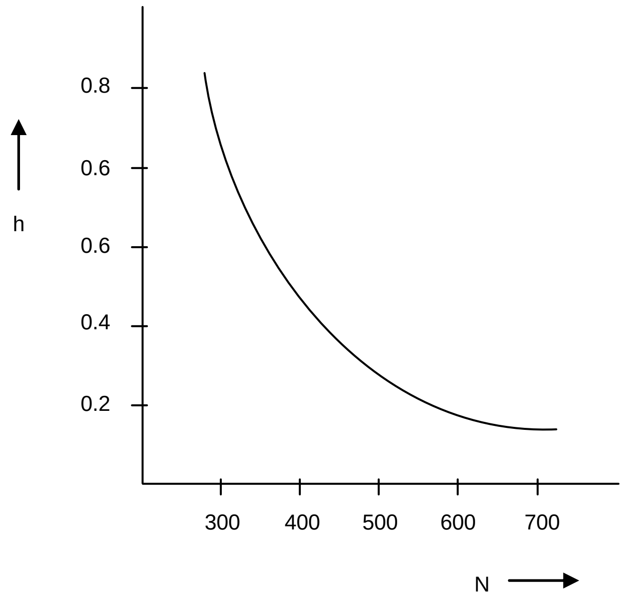


Figure 5.3 : Graph between Height and Speed

SAQ 2

Why watt governor is very rarely used? Give reasons.

5.3.2 Porter Governor

A schematic diagram of the porter governor is shown in Figure 5.4(a). There are two sets of arms. The top arms OA and OB connect balls to the hinge O. The hinge may be on the spindle or slightly away. The lower arms support dead weight and connect balls also. All of them rotate with the spindle. We can consider one-half of governor for equilibrium.

Let w be the weight of the ball,

T_1 and T_2 be tension in upper and lower arms, respectively,

F_c be the centrifugal force,

r be the radius of rotation of the ball from axis, and

I is the instantaneous centre of the lower arm.

Taking moment of all forces acting on the ball about I and neglecting friction at the sleeve, we get

$$F_C \times AD - w \times ID - \frac{W}{2} IC = 0$$

or $$F_C = \frac{w\,ID}{AD} + \frac{W}{2}\left(\frac{ID + DC}{AD}\right)$$

or $$F_C = w \tan \alpha + \frac{W}{2} (\tan \alpha + \tan \beta)$$

$$F_C = \frac{w}{g}\, \omega^2\, r$$

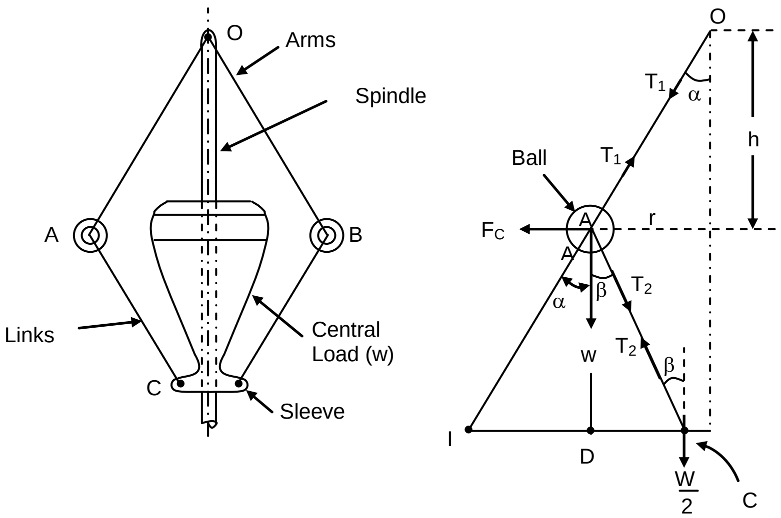
$\therefore$ $$\frac{w}{g}\, \omega^2 r = w \tan \alpha \left\{1 + \frac{W}{2w}\left(1 + \frac{\tan \beta}{\tan \alpha}\right)\right\}$$

or $$\omega^2 = \frac{g}{r} \tan \alpha \left\{1 + \frac{W}{2w}(1 + K)\right\} \qquad \ldots (5.3)$$

where $$K = \frac{\tan \beta}{\tan \alpha}$$

$\because$ $$\tan \alpha = \frac{r}{h}$$

$\therefore$ $$\omega^2 = \frac{g}{h}\left\{1 + \frac{W}{2w}(1 + K)\right\} \qquad \ldots (5.4)$$



(a) (b)

Figure 5.4 : Porter Governor

If friction at the sleeve is f, the force at the sleeve should be replaced by $W + f$ for rising and by $(W - f)$ for falling speed as friction apposes the motion of sleeve. Therefore, if the friction at the sleeve is to be considered, W should be replaced by $(W \pm f)$. The expression in Eq. (5.4) becomes

$$\omega^2 = \frac{g}{h}\left\{1 + \frac{(W \pm f)}{2w}(1+K)\right\} \qquad \ldots (5.5)$$

SAQ 3

In which respect Porter governor is better than Watt governor?

5.4 SPRING CONTROLLED CENTRIFUGAL GOVERNORS

In these governors springs are used to counteract the centrifugal force. They can be designed to operate at high speeds. They are comparatively smaller in size. Their speed range can be changed by changing the initial setting of the spring. They can work with inclined axis of rotation also. These governors may be very suitable for IC engines, etc.

The most commonly used spring controlled centrifugal governors are :

(a) Hartnell governor

(b) Wilson-Hartnell governor

(c) Hartung governor

5.4.1 Hartnell Governor

The Hartnell governor is shown in Figure 5.5. The two bell crank levers have been provided which can have rotating motion about fulcrums O and O'. One end of each bell crank lever carries a ball and a roller at the end of other arm. The rollers make contact with the sleeve. The frame is connected to the spindle. A helical spring is mounted around the spindle between frame and sleeve. With the rotation of the spindle, all these parts rotate.

With the increase of speed, the radius of rotation of the balls increases and the rollers lift the sleeve against the spring force. With the decrease in speed, the sleeve moves downwards. The movement of the sleeve are transferred to the throttle of the engine through linkages.

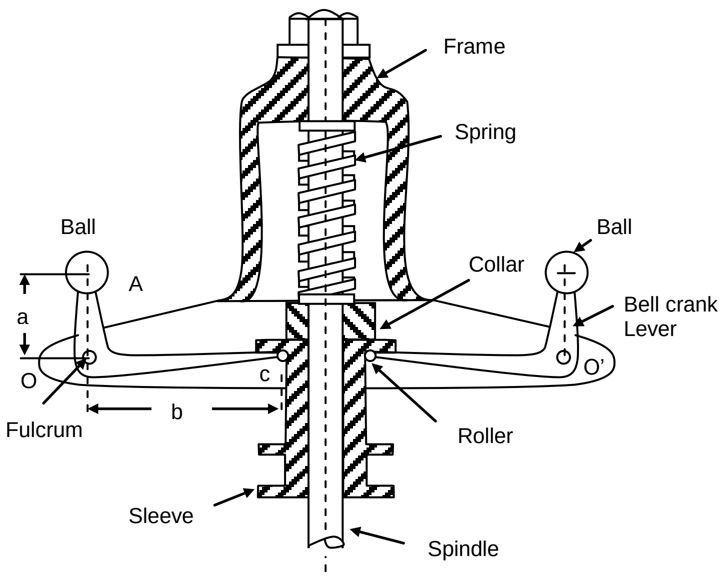


Figure 5.5 : Hartnell Governor

Let r_1 = Minimum radius of rotation of ball centre from spindle axis, in m,

r_2 = Maximum radius of rotation of ball centre from spindle axis, in m,

S_1 = Spring force exerted on sleeve at minimum radius, in N,

S_2 = Spring force exerted on sleeve at maximum radius, in N,

m = Mass of each ball, in kg,

M = Mass of sleeve, in kg,

N_1 = Minimum speed of governor at minimum radius, in rpm,

N_2 = Maximum speed of governor at maximum radius, in rpm,

ω_1 and ω_2 = Corresponding minimum and maximum angular velocities, in r/s,

$(F_C)_1$ = Centrifugal force corresponding to minimum speed $= m \times \omega_1^2 \times r_1$,

$(F_C)_2$ = Centrifugal force corresponding to maximum speed $= m \times \omega_2^2 \times r_2$,

s = Stiffness of spring or the force required to compress the spring by one m,

r = Distance of fulcrum O from the governor axis or radius of rotation,

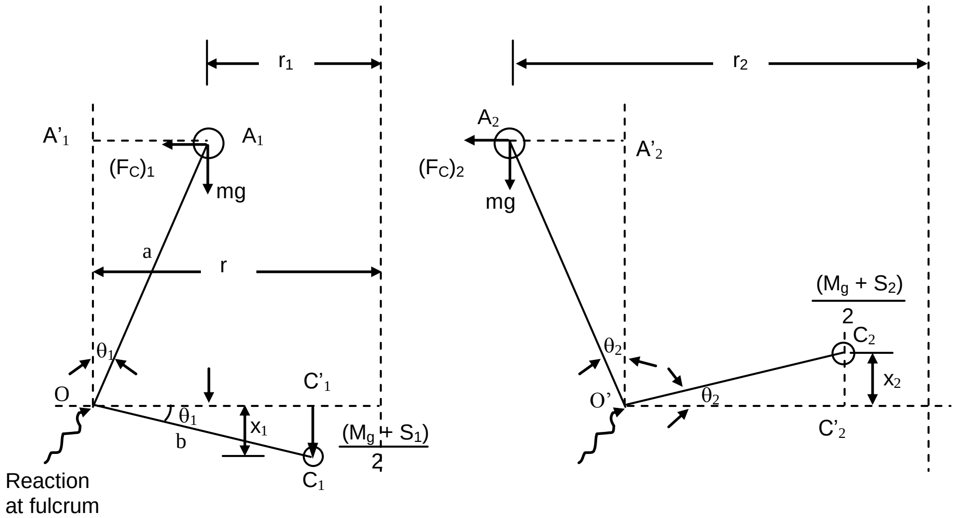
a = Length of ball arm of bell-crank lever, i.e. distance OA, and

b = Length of sleeve arm of bell-crank lever, i.e. distance OC.

Considering the position of the ball at radius 'r_1', as shown in Figure 5.6(a) and taking moments of all the forces about O

$$M_0 = (F_C)_1\, a \cos \theta_1 - mg\, a \sin \theta_1 - \frac{(Mg + S_1)}{2}\, b \cos \theta_1 = 0$$

or $$(F_C)_1 = mg \tan \theta_1 + \frac{(Mg + S_1)}{2} \left(\frac{b}{a}\right) \qquad \text{. . . (5.9)}$$



(a) (b)

Figure 5.6

Considering the position of the ball at radius 'r_2' as shown in Figure 5.6(b) and taking the moments of all the forces about O'

$$M_0' = (F_C)_2\, a \cos \theta_2 + mg\, a \sin \theta_2 - \frac{(Mg + S_2)}{2}\, b \cos \theta_2$$

or $$(F_C)_2 = \frac{(Mg + S_2)}{2} \left(\frac{b}{a}\right) - mg \tan \theta_2 \qquad \text{. . . (5.10)}$$

If θ_1 and θ_2 are very small and mass of the ball is negligible as compared to the spring force, the terms $mg \tan \theta_1$ and $mg \tan \theta_2$ may be ignored.

$$(F_C)_1 = \frac{(Mg + S_1)}{2} \left(\frac{b}{a} \right) \quad \ldots (5.11)$$

and $$(F_C)_2 = \frac{(Mg + S_2)}{2} \left(\frac{b}{a} \right) \quad \ldots (5.12)$$

$$\therefore \quad (F_C)_2 - (F_C)_1 = \frac{(S_2 - S_1)}{2} \left(\frac{b}{a} \right)$$

$$\text{Total lift } = (x_1 + x_2) \square (b\,\theta_1 + b\,\theta_2)$$

$$= b\,(\theta_1 + \theta_2)$$

$$= b \left(\frac{(r - r_1)}{a} + \frac{(r_2 - r)}{a} \right) = \frac{b}{a} (r_2 - r_1)$$

$$S_2 - S_1 = \text{Total lift} \times s = \frac{b}{a} (r_2 - r_1)\, s$$

$$\therefore \quad (F_C)_2 - (F_C)_1 = \left(\frac{b}{a} \right)^2 \frac{(r_2 - r_1)}{2}\, s$$

or stiffness of spring 's' $$= 2 \left(\frac{a}{b} \right)^2 \frac{(F_C)_2 - (F_C)_1}{(r_2 - r_1)} \quad \ldots (5.13)$$

For ball radius 'r'

$$s = 2 \left(\frac{a}{b} \right)^2 \frac{F_C - (F_C)_1}{r - r_1} = 2 \left(\frac{a}{b} \right)^2 \left\{ \frac{(F_C)_2 - (F_C)_1}{(r_2 - r_1)} \right\}$$

$$\therefore \quad F_C = (F_C)_1 + \frac{(r - r_1)}{(r_2 - r_1)} \{(F_C)_2 - (F_C)_1\} \quad \ldots (5.14)$$

SAQ 4

For IC engines, which type of governor you will prefer whether dead weight type or spring controlled type? Give reasons.

5.5 GOVERNOR EFFORT AND POWER

Governor effort and power can be used to compare the effectiveness of different type of governors.

Governor Effort

It is defined as the mean force exerted on the sleeve during a given change in speed.

When governor speed is constant the net force at the sleeve is zero. When governor speed increases, there will be a net force on the sleeve to move it upwards and sleeve starts moving to the new equilibrium position where net force becomes zero.

Governor Power

It is defined as the work done at the sleeve for a given change in speed. Therefore,

Power of governor = Governor effort × Displacement of sleeve

5.5.1 Determination of Governor Effort and Power

The effort and power of a Porter governor has been determined. The same principle can be used for any other type of governor also.

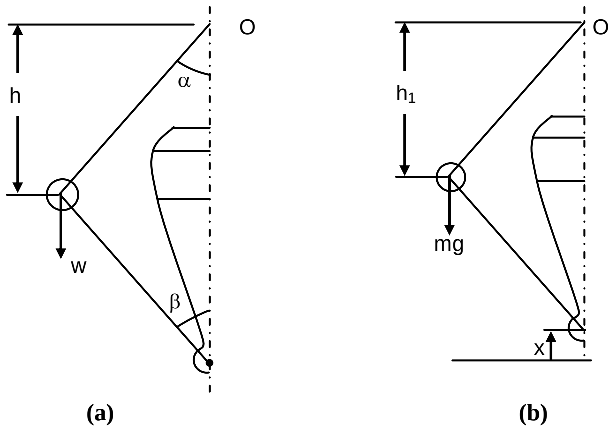


Figure 5.7

Figure 5.7 shows the two positions of a Porter governor.

Let N = Equilibrium speed corresponding to configuration shown in Figure 5.7(a),

W = Weight of sleeve in N,

h = Height of governor corresponding to speed N, and

c = A factor which when multiplied to equilibrium speed, gives the increase in speed.

∴ Increased speed = Equilibrium speed + Increase of speed,

$$= N + c\,.\,N = (1 + c)\,N, \text{ and} \quad \ldots (5.15)$$

h_1 = Height of governor corresponding to increased speed $(1 + c)\,N$.

The equilibrium position of the governor for the increased speed is shown in Figure 5.7(b). In order to prevent the sleeve from rising when the increase of speed takes place, a downward force will have to be exerted on the sleeve.

Let W_1 = New weight of sleeve so that the rising of sleeve is prevented when the speed is $(1 + c)\,N$. This means that W_1 is the weight of sleeve when height of governor is h.

∴ Downward force to be applied when the rising of sleeve is to be prevented when speed increases from N to $(1 + c)\,N = W_1 - W$.

When speed is N rpm and let the angles α and β are equal so that $K = 1$, the height h is given by equation

$$h = \left(\frac{w + W}{w}\right) \times \frac{g}{\left(\frac{2\pi N}{60}\right)^2} \quad \ldots (5.16)$$

If the speed increases to $(1 + c)\,N$ and height remains the same by increasing the load on sleeve

$$h = \left(\frac{w + W_1}{w}\right) \times \frac{g}{\left\{\frac{2\pi\,(1 + c)\,N}{60}\right\}^2} \quad \ldots (5.17)$$

Equating the two values of h given by above equations, we get

$$w + W = \frac{\{(w + W_1)\}}{(1 + c)^2}$$

$$(w + W)\,(1 + c)^2 = w + W_1$$

$$W_1 = (w + W)\,(1 + c)^2 - w$$

$$(W_1 - W) = (w + W)\,(1 + c)^2 - (w + W)$$

$$= (w + W)\,\{(1 + c)^2 - 1\}$$

$$\approx 2c\,(w + W) \quad \text{If } c \text{ is very small} \qquad \ldots (5.18)$$

But $W_1 - W$ is the downward force which must be applied in order to prevent the sleeve from rising when the increase of speed takes place. This is also the force exerted by the governor on the sleeve when the speed changes from N to $(1 + c)\,N$. As the sleeve rises to the new equilibrium position as shown in Figure 5.7(b), this force gradually diminishes to zero. The mean force P exerted on the sleeve during the change of speed from N to $(1 + c)\,N$ is therefore given by

$$P = \frac{W_1 - W}{2} \approx c\,(w + W) \qquad \ldots (5.19)$$

This is the governor effort.

If weight on the sleeve is not increased

$$h_1 = \left(\frac{w + W}{w}\right) \frac{g}{\left\{\frac{2\pi\,(1 + c)\,N}{60}\right\}^2} \qquad \ldots (5.20)$$

$$h - h_1 = 2x$$

$$\frac{h}{h_1} = (1 + c)^2$$

$\therefore$
$$\frac{h}{h_1} - 1 = (1 + c)^2 - 1 \approx 2c$$

or
$$\frac{h - h_1}{h_1} = 2c$$

or
$$\frac{2x}{h_1} = 2c$$

or
$$x = c\,h_1$$

$\therefore$ Governor power $= Px = c^2\,h_1\,(w + W)$. $\qquad \ldots (5.21)$

5.6 CHARACTERISTICS OF GOVERNORS

Different governors can be compared on the basis of following characteristics :

Stability

A governor is said to be stable when there is one radius of rotation of the balls for each speed which is within the speed range of the governor.

Sensitiveness

The sensitiveness can be defined under the two situations :

(a) When the governor is considered as a single entity.

(b) When the governor is fitted in the prime mover and it is treated as part of prime mover.

(a) A governor is said to be sensitive when there is larger displacement of the sleeve due to a fractional change in speed. Smaller the change in speed of the governor for a given displacement of the sleeve, the governor will be more sensitive.

$$\therefore \qquad \text{Sensitiveness} = \frac{N}{N_1 - N_2} \qquad \ldots (5.22)$$

(b) The smaller the change in speed from no load to the full load, the more sensitive the governor will be. According to this definition, the sensitiveness of the governor shall be determined by the ratio of speed range to the mean speed. The smaller the ratio more sensitive the governor will be

$$\therefore \qquad \text{Sensitiveness} = \frac{N_2 - N_1}{N} = \frac{2(N_2 - N_1)}{(N_2 + N_1)} \qquad \ldots (5.23)$$

where $N_2 - N_1$ = Speed range from no load to full load.

Isochronism

A governor is said to be isochronous if equilibrium speed is constant for all the radii of rotation in the working range. Therefore, for an isochronous governor the speed range is zero and this type of governor shall maintain constant speed.

Hunting

Whenever there is change in speed due to the change in load on the engine, the sleeve moves towards the new position but because of inertia if overshoots the desired position. Sleeve then moves back but again overshoots the desired position due to inertia. This results in setting up of oscillations in engine speed. If the frequency of fluctuations in engine speed coincides with the natural frequency of oscillations of the governor, this results in increase of amplitude of oscillations due to resonance. The governor, then, tends to intensity the speed variation instead of controlling it. This phenomenon is known as hunting of the governor. Higher the sensitiveness of the governor, the problem of hunting becomes more acute.

5.7 CONTROLLING FORCE AND STABILITY OF SPRING CONTROLLED GOVERNORS

The resultant external force which controls the movement of the ball and acts along the radial line towards the axis is called controlling force. This force acts at the centre of the ball. It is equal and acts opposite to the direction of centrifugal force.

The controlling force 'F' = $m \omega^2 r$.

Or
$$\frac{F}{r} = m \left(\frac{2\pi N}{60}\right)^2$$

For controlling force diagram in which 'F' is plotted against radius 'r', $\frac{F}{r}$ represents slope of the curve.

i.e.
$$\frac{F}{r} = \tan \phi \propto N^2 \qquad \ldots (5.24)$$

Therefore, for a stable governor slope in controlling force diagram should increase with the increase in speed.

Stability of Spring-controlled Governors

Figure 5.8 shows the controlling force curves for stable, isochronous and unstable spring controlled governors. The controlling force curve is approximately straight line for spring controlled governors. As controlling force curve represents the variation of controlling force 'F' with radius of rotation 'r', hence, straight line equation can be,

$$F = ar + b; \quad F = ar \quad \text{or} \quad F = ar - b \qquad \ldots (5.25)$$

where a and b are constants. In the above equation b may be +ve, or –ve or zero.

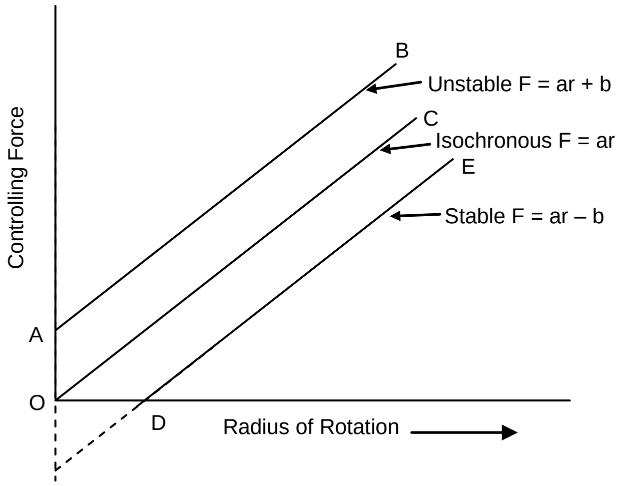


Figure 5.8 : Stability of Spring Controlled Governors

These three cases are as follows :

(a) We know that for a stable governor, the ratio $\frac{F}{r}$ must increase as r increases. Hence the controlling force curve DE for a stable governor must intersect the controlling force axis (i.e. y-axis) below the origin, when produced. Then the equation of the curve will be of the form

$$F = a \, . \, r - b \quad \text{or} \quad \frac{F}{r} = a - \frac{b}{r} \qquad \ldots (5.26)$$

As r increases $\frac{F}{r}$ increase and thereby tan ϕ increases. Therefore, this equation represents stable governor.

(b) If b in the above equation is zero then the controlling force curve OC will pass through the origin. The ratio $\frac{F}{r}$ will be constant for all radius of rotation and hence the governor will become isochronous. Hence for isochronous, the equation will be

$$F = ar \quad \text{or} \quad \frac{F}{r} = a = \text{constant} \qquad \ldots (5.27)$$

(c) If b is positive, then controlling force curve AB will intersect the controlling force axis (i.e. y-axis) above the origin. The equation of the curve will be

$$F = ar + b \quad \text{or} \quad \frac{F}{r} = a + \frac{b}{r} \qquad \ldots (5.28)$$

As r increases, speed increases, $\frac{F}{r}$ or tan ϕ reduces. Hence this equation cannot represent stable governor but unstable governor.

5.8 INSENSITIVENESS IN THE GOVERNORS

The friction force at the sleeve gives rise to the insensitiveness in the governor. At any given radius there will be two different speeds one being when sleeve moves up and other when sleeve moves down. Figure 5.9 shows the controlling force diagram for such a governor.

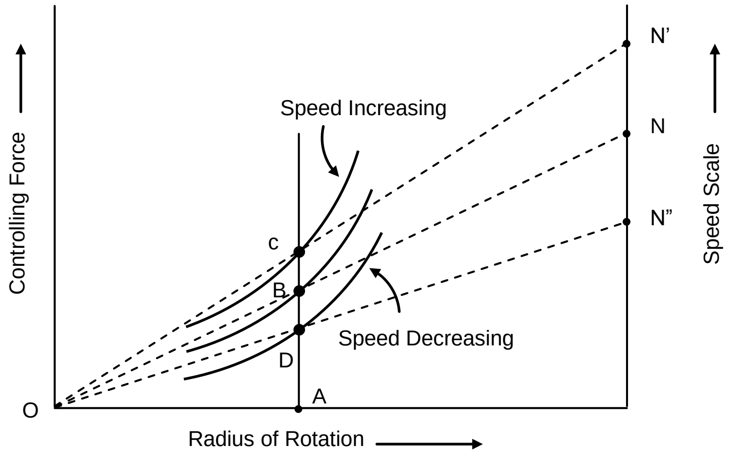


Figure 5.9 : Insensitiveness in the Governors

The corresponding three values of speeds for the same radius *OA* are :

(a) The speed *N* when there is no friction.

(b) The speed N' when speed is increasing or sleeve is on the verge of moving up, and

(c) The speed N'' when speed is decreasing or sleeve on the verge of moving down.

This means that, when radius is *OA*, the speed of rotation may vary between the limits N'' and N', without causing any displacement of the governor sleeve. The governor is said to be insensitive over this range of speed. Therefore,

$$\therefore \quad \text{Coefficient of insensitiveness } = \left(\frac{N' - N''}{N} \right) \qquad \ldots (5.29)$$

Example 5.1

The arms of a Porter governor are 25 cm long and pivoted on the governor axis. The mass of each ball is 5 kg and mass on central load of the sleeve is 30 kg. The radius of rotation of balls is 15 cm when the sleeve begins to rise and reaches a value of 20 cm for the maximum speed. Determine speed range.

Solution

Given data : Ball weight 'w' = 5 g N

Central load 'W' = 30 g N

Arm length 'l' = 25 cm = 0.25 m

Minimum radius 'r_1' = 15 cm = 0.15 m

Maximum radius 'r_2' = 20 cm = 0.2 m

$$\text{Height } 'h_1' = \sqrt{l^2 - r_1^2} = \sqrt{0.25^2 - 0.15^2} = 0.2 \text{ m}$$

For $k = 1$.

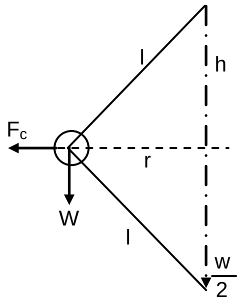


Figure 5.10 : Figure for Example 5.1

Substituting values in Eq. (5.4)

$$\omega_1^2 = \frac{g}{0.2}\left\{1 + \frac{W}{2w}(1+1)\right\}$$

$$= \frac{9.81}{0.2}\left\{1 + \frac{30\,g}{5\,g}\right\}$$

$$\omega_1 = 18.5297 \text{ r/s} \quad \text{or} \quad N_1 = 176.9 \text{ rpm}$$

$$\text{Height } h_2 = \sqrt{0.25^2 - 0.2^2} = 0.15 \text{ m}$$

$$\therefore \qquad \omega_2^2 = \frac{9.81}{0.15}\left\{1 + \frac{30\,g}{5\,g}\right\}$$

$$\therefore \qquad \omega_2 = 29.396 \text{ r/s} \quad \text{or} \quad N_2 = 204.32 \text{ rpm}$$

Speed range = $N_2 - N_1$ = 204.32 – 176.9 = 27.42 rpm.

Example 5.2

In a Hartnell governor the radius of rotation is 7 cm when speed is 500 rpm. At this speed, ball arm is normal and sleeve is at mid position. The sleeve movement is 2 cm with ± 5% of change in speed. The mass of sleeve is 6 kg and friction is equivalent to 25 N at the sleeve. The mass of the ball is 2 kg. If ball arm and sleeve arms are equal, find,

(a) Spring rate,

(b) Initial compression in the spring, and

(c) Governor effort and power for 1% change in the speed if there is no friction.

Solution

Sleeve mass ‘M’ = 6 kg

Friction force ‘f’ = 25 N

Ball mass ‘m’ = 2 kg

$\because \qquad a = b$

Minimum radius r_1 = 7 cm – 1 = 6 cm

Maximum radius r_2 = 7 cm + 1 = 8 cm

$$\omega = \frac{2\pi \times 500}{60} = 52.36 \text{ r/s}$$

Maximum speed = 10.05 ω = 1.05 × 52.36 = 54.98 r/s

Minimum speed = 0.95 ω = 0.95 × 52.36 = 49.74 r/s

Neglecting the effect of obliquity of arms.

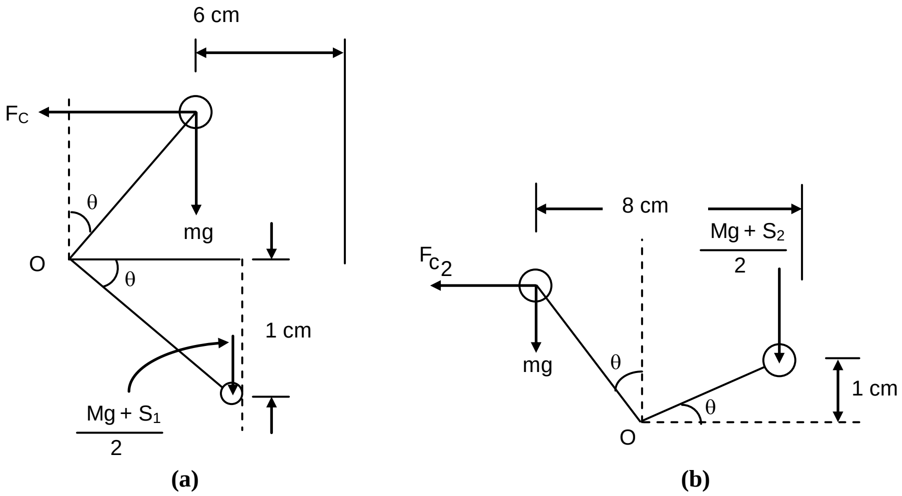


Figure 5.11 : Figure for Example 5.2

At Minimum Radius

$$F_{C_1} \times a = b\left(\frac{Mg + S_1 - f}{2}\right) \quad or \quad 2F_{C_1} = Mg + S_1 - f$$

$$F_{c1} = m\,\omega_1^2\,r_1$$

$\therefore$ $$2 \times (49.74)^2 \times 0.06 \times 2 = 6g + S_1 - 25$$

$$593.78 = 58.86 + S_1 - 25$$

Or $$S_1 = 559.92\ N$$

At Maximum Radius

$$2F_{C_2} = Mg + S_2 + f$$

$$F_{c2} = m\,\omega_2^2\,r_2$$

$\therefore$ $$2 \times (54.98)^2 \times 0.08 \times 2 = 6g + S_2 + 25$$

Or $$S_2 = 883.44\ N$$

$\therefore$ Stiffness 's' $= \dfrac{S_2 - S_1}{x}$

$$= \frac{883.44 - 559.92}{0.02}$$

Or $s = 16175.81$ N/m

Initial compression $= S_1 = \dfrac{559.92}{16175.81}$

$= 0.035$ m or 3.5 cm

Governor Effort and Power

$$F_C = \frac{Mg + S_2 \pm f}{2}$$

Increased speed = 1.01 ω = 1.01 × 52.36 = 52.88 r/s

At $r = 0.07$; $2 \times 2 \times (52.36)^2 \times 0.07 = 6\,g + S$

At increased speed, $2 \times 2 \times (52.88)^2 \times 0.07 = 6\,g + 2\,P + S$

where P is governor effort.

$\therefore \quad 2P = 2 \times 2 \times 0.07 \{(52.88)^2 - (52.36)^2\}$

Or $\quad P = 7.66$ N

Let the spring force corresponding to speed 52.88 r/s be S'.

$\therefore \quad 2 \times 2 \times (52.88)^2 \times 0.07 = 6g + S'$

$\therefore \quad (S' - S) = 2 \times 2 \times 0.07 \times \{(52.88)^2 - (52.36)^2\}$

$= 15.32$ N

Sleeve lift for 1% change $= \dfrac{15.32}{s}$

$= \dfrac{15.32}{16175.81} = 9.47 \times 10^{-4}$ m

$\therefore$ Governor power $= 7.66 \times 9.47 \times 10^{-4}$

$= 7.25 \times 10^{-3}$ Nm

Example 5.3

The controlling force diagram of a spring controlled governor is a straight line. The weight of each governor ball is 40 N. The extreme radii of rotation of balls are 10 cm and 17.5 cm. The corresponding controlling forces at these radii are 205 N and 400 N. Determine :

(a) the extreme equilibrium speeds of the governor, and

(b) the equilibrium speed and the coefficient of insensitivenss at a radius of 15 cm. The friction of the mechanism is equivalent of 2.5 N at each ball.

Solution

Weight of each ball 'w' = 40 N

$r_1 = 10$ cm and $r_2 = 17.5$ cm

$F_{C_1} = 205$ N and $F_{C_2} = 400$ N

Let $\quad F_C = ar + b$

when $\quad r_1 = 10 \text{ cm} = 0.1 \text{ m}$ and $F_{C_1} = 205$ N

$205 = b + 0.1\, a$

when $\quad r_2 = 17.5 \text{ cm} = 0.175 \text{ m}$ and $F_{C_2} = 400$ N

$400 = b + 0.175\, a$

$\therefore \quad 195 = 0.075\, a \;\Rightarrow\; a = 2600$

$\therefore \quad b = 205 - 0.1 \times 2600 = -55$

$\therefore \quad F_C = -55 + 2600\, r$

(a) For $\quad F_C = 205; \quad \dfrac{40}{g}\left(\dfrac{2\pi N_1}{60}\right)^2 \times 0.1 = 205$ N

Or $\quad N_1 = 214.1$ rpm

For $\quad F_C = 400; \quad r = 0.175$ m

$\therefore \quad \dfrac{40}{g}\left(\dfrac{2\pi N_2}{60}\right)^2 \times 0.175 = 400$

Or $\quad N_2 = 226.1$ rpm

(b) $F_C = k\,N^2$

At radius r = 15 cm

$$F_C + f_b = k\,N'^2$$

$$F_C - f_b = k\,N''^2$$

$$(F_C + f_b) - (F_C - f_b) = k\,(N'^2 - N''^2)$$

Or $\quad 2\,f_b = k\,(N' - N'')\,(N' + N'')$

$$= 2k\,(N' - N'')\,N$$

$$\frac{2\,f_b}{F_C} = \frac{2k\,(N' - N'')\,N}{k\,N^2} = \frac{2\,(N' - N'')}{N}$$

$\therefore$ Coefficient of insensitiveness $= \dfrac{(N' - N'')}{N} = \dfrac{1}{2} \times \dfrac{2\,f_b}{F_C} = \dfrac{f_b}{F_C}$

At $\quad r = 0.15$ m

$$F_C = -\,55 + 2600 \times 0.15 = 335 \text{ N}$$

$\therefore$ Coefficient of insensitiveness $= \dfrac{2.5}{335} = 7.46 \times 10^{-3}$ Or 0.746%.

5.9 SUMMARY

The governors are control mechanisms and they work on the principle of feedback control. Their basic function is to control the speed within limits when the load on the prime mover changes. They have no control over the change is speed within the cycle. The speed control within the cycle is done by the flywheel.

The governors are classified in three main categories that is centrifugal governors, inertial governor and pickering governor. The use of the two later governors is very limited and in most of the cases centrifugal governors are used. The centrifugal governors are classified into two main categories, gravity controlled type and spring loaded type.

The gravity controlled type of governors are larger in size and require more space as compared to the spring controlled governors. This type of governors are two, i.e. Porter governor and Proell governor. The spring controlled governors are : Hartnel governor, Wilson-Hartnell governor and Hartung governor.

For comparing different type of governors, effort and power is used. They determine whether a particular type of governor is suitable for a given situation or not. To categorise a governor the characteristics can be used. It can be determined whether a governor is stable or isochronous or it is prone to hunting. The friction at the sleeve gives rise to the insensitiveness in the governor. At any particular radius, there shall be two speeds due to the friction. Therefore, it is most desirable that the friction should be as low as possible.

The stability of a spring controlled governor can be determined by drawing controlling force diagram which should have intercept on the negative side of Y-axis.

5.10 KEY WORDS

Watt Governor : It is a type of governor which does not have load on the sleeve.

Porter Governor : This is a type of governor which has dead weight at the sleeve and balls are mounted at the hinge.

Hartnell Governor	**:**	It is a spring controlled governor in which balls are mounted on the bell crank lever and sleeve is loaded by spring force.
Governor Effort	**:**	It is the mean force exerted on the sleeve during a given change of speed.
Governor Power	**:**	It is defined as the work done at the sleeve for a given change in speed.
Hunting of Governor	**:**	It can occur in governor when the fluctuations in engine speed coincides the natural frequency of oscillations of the governor. In that case governor intensifies the speed variation instead of controlling it.
Controlling Force	**:**	It is the resultant external force which controls the movement of the ball and acts along the radial line towards the axis.

5.11 ANSWERS TO SAQs

Refer the preceding text for all the Answers to SAQs.